

### Problem Set II

Due: Tuesday, 7 September 2010

1. (Problem 3.8.4, p. 136)

Determine whether the vectors

$$\begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 6 \\ 14 \\ 5 \end{pmatrix}$$

are linearly dependent or independent. If they are linearly dependent, find a maximal linearly independent subset and write each of the given vectors as a linear combination of those linearly independent ones.

2. (Problems 3.8.17 and 27, pp. 136–137)

a. Solve the following system of homogeneous linear equations by row reducing the matrix:

$$\begin{aligned} x - 2y + 3z &= 0 \\ x + 4y - 6z &= 0 \\ 2x + 2y - 3z &= 0 \end{aligned}$$

b. Solve the following system of inhomogeneous linear equations and write the solution in vector form, as in (8.11) and (8.13) in the book:

$$\begin{aligned} x - y + 2z &= 3 \\ -2x + 2y - z &= 0 \\ 4x - 4y + 5z &= 6 \end{aligned}$$

3. (Problems 3.9.8 and 18, pp. 141–142)

a. Prove that  $(AB)^\dagger = B^\dagger A^\dagger$  for all matrices  $A$  and  $B$  (whose product exists).

*Hint:* See Eq. (9.10) from the book.

b. Extend the previous result to show that  $(AB \cdots CD)^\dagger = D^\dagger C^\dagger \cdots B^\dagger A^\dagger$  for any product of  $n$  matrices (whose product exists).

*Hint:* This will have to be a proof by induction.

c. If  $A$  and  $B$  are Hermitian matrices, show that their commutator is anti-Hermitian.

4. (Problems 3.11.13, 16 and 24, p. 159)

Find the eigenvectors and eigenvalues of the following matrices:

$$A := \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}, \quad B := \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & -1 \end{pmatrix} \quad \text{and} \quad C := \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}.$$

How many eigenspaces does each matrix have?

5. (Problem 3.11.61, p. 162)

At the end of Section 3.9, the book proved that if  $H$  is a Hermitian matrix, then the matrix  $e^{iH}$  is unitary. Give another proof of this by writing  $H = C D C^{-1}$ , where  $C$  is unitary and  $D$  is diagonal and real. Show that  $e^{iD}$  is unitary and that  $e^{iH}$  is a product of three unitary matrices. Finally, show that such a product of unitary matrices is unitary.

6. (Problems 3.12.17 and 18, p. 172)

Find the characteristic frequencies and characteristic modes of vibration for systems of masses and springs as in Figure 12.1 and Examples 3, 4 and 6 from Section 3.12 of the book for the following arrays of springs and masses:

a.  $3k, 3m, 2k, 4m, 2k$ ,

b.  $2k, m, k, 5m, 10k$ .

In each case, the springs and masses are arrayed in the order shown, from left to right, between two fixed walls. See Figure 12.1 for an example.

7. (Problem 3.15.34, p. 187)

The Pauli spin matrices are defined by

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

a. Calculate the exponentials  $e^{\theta\sigma_x}$ ,  $e^{\theta\sigma_y}$  and  $e^{\theta\sigma_z}$  for  $\theta$  real. Show that each is unitary.

b. Calculate the exponential  $e^{\sigma_x + \sigma_z}$  and show that it is not equal to the product  $e^{\sigma_x} e^{\sigma_z}$ .