Physics (PHZ) 3113 | Florida Atlantic University
Mathematical Physics | Fall, 2009

Review Problems III

Recommended Reading: Chow, pp. 1–13; Boas, Sections 3.4–3.5.

1. (Chow 1.2) Show that there is a unique plane in three-dimensional space containing the vectors $\mathbf{A} := (2, -6, -3)$ and $\mathbf{B} := (4, 3, -1)$, and find a unit vector normal to it.

2. (Chow 1.7) Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be three-dimensional vectors.

   a. Show that the three vectors are co-planar if and only if

   $$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = 0.$$

   b. Find a necessary and sufficient condition for an arbitrary vector $\mathbf{x}$ to lie in the plane containing the points at the tips of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$.

3. (Boas 3.4.4, 6 and 28) Prove the following theorems of plane geometry using vector algebra.

   a. The line segment joining the midpoints of two sides of any triangle is parallel to the third side and half its length.

   b. The lines joining the midpoints of opposite sides of a quadrilateral (any figure with four sides of arbitrary length with arbitrary angles between them) bisect each other. *Hint*: If $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are vectors along the four sides of the quadrilateral, what is their sum?

   c. The diagonals of a rhombus (a figure with four sides of equal length that in general do not meet at right angles) bisect one another and are perpendicular.

4. (Boas 3.4.12) Find the angle between the vectors $\mathbf{A} := -2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{B} := 2\mathbf{i} - 2\mathbf{j}$.

5. (Boas 3.4.24) Let $\mathbf{A}$ be a fixed, but arbitrary, vector in three dimensions.

   a. If $\mathbf{A} \cdot \mathbf{B} = 0$, does it follow that $\mathbf{B} = \mathbf{0}$?

   b. If $\mathbf{A} \times \mathbf{B} = \mathbf{0}$, does it follow that $\mathbf{B} = \mathbf{0}$?

   c. If both $\mathbf{A} \cdot \mathbf{B} = 0$ and $\mathbf{A} \times \mathbf{B} = \mathbf{0}$, does it follow that $\mathbf{B} = \mathbf{0}$?

   Give a proof or a counterexample in each case.

6. (Boas 3.5.8, 12, 15 and 18) Give the symmetric equations (5.6 or 7) and parametric equations (5.8) of each line, and the equation (5.10) of each plane, described below.

   a. The line passing through $(0, -2, 4)$ and $(3, -2, -1)$.

   b. The line passing through $(5, -4, 2)$ and parallel to the line $\mathbf{x} = \mathbf{i} - \mathbf{j} + (5\mathbf{i} - 2\mathbf{j} + \mathbf{k})t$.

   c. The plane containing the origin and both points of part (a).

   d. The plane containing the two parallel lines of part (b).

7. (Boas 3.5.22) Find the (acute) angle between the planes $2x - y - z = 4$ and $3x - 2y - 6z = 7$.

8. (Boas 3.5.30) Find the distance from the origin to the plane $2x - 2y - 6z = 7$.

9. (Boas 3.5.45) A particle travels along the line

$$\frac{x-3}{2} = \frac{y+1}{-2} = z - 1.$$

   Write the equation of its path in the form $\mathbf{r}(\lambda) = \mathbf{r}_0 + \mathbf{a}\lambda$. Find the distance of closest approach of the particle to the origin (i.e., the distance from the origin to the line). Show that the point of closest approach occurs at

$$\lambda = \frac{-\mathbf{r}_0 \cdot \mathbf{a}}{\|\mathbf{a}\|^2}.$$

   Use this value to check your result for the distance of closest approach. Under what circumstances is it permissible to replace the parameter λ with the Newtonian time t?

10. The parametric equation for the straight line passing through the point $\mathbf{x}_0$ and parallel to the vector $\mathbf{a}$ is given by Boas (3.5.8) as

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{a}t,$$

   where t is an arbitrary real parameter that takes arbitrary values. Develop a similar parametric equation for the plane passing through $\mathbf{x}_0$ and parallel to the plane containing the vectors $\mathbf{a}$ and $\mathbf{b}$.

   Hint: Because the plane in question is *two*-dimensional, you will need *two* real parameters u and v to specify its points.