Physics (PHZ) 3113 Florida Atlantic University
Mathematical Physics Fall, 2009

Review Problems IV

Recommended Reading: Chow, pp. 15–27.

1. (Chow 1.12)

   a. Find a unit vector normal to the surface $x^2 + y^2 - z = 1$ at the point $(1, 1, 1)$.

   b. Find the directional derivative of $\phi(x, y, z) = x^2yz + 4xz^3$ at the points $(1, -2, -1)$ in the direction $\hat{\mathbf{x}} - 2\hat{\mathbf{y}} + 2\hat{\mathbf{z}}$.

2. (Chow 1.14) Consider the ellipse $r_1 + r_2 = a$, with a a constant, that is shown in Figure 1.23 of the text. Prove that the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ from either focus to a given point on the ellipse make equal angles with the tangent to the ellipse at that point.

3. (based on Chow 1.17)

   a. Show that the divergence of an inverse-square force field in three dimensions is zero,

   $$\boldsymbol{\nabla} \cdot \mathbf{F} := \boldsymbol{\nabla} \cdot \frac{\mathbf{r}}{r^3} = 0,$$

   except possibly at the origin.

   b. Show that the flux of $\mathbf{F}$ through a sphere of radius a about the origin is 4π. Use Gauss' theorem to show that the divergence of $\mathbf{F}$ cannot vanish at the origin.

   c. If f is a differentiable function and $\mathbf{A}$ is a differentiable vector field, then show that

   $$\boldsymbol{\nabla} \cdot (f\mathbf{A}) = \mathbf{A} \cdot \boldsymbol{\nabla} f + f\, \boldsymbol{\nabla} \cdot \mathbf{A}.$$

   d. Calculate the divergence of a ***radial*** vector field (i.e., everywhere parallel to $\mathbf{r}$) in three dimensions. Under what conditions does the divergence vanish at the origin?

4. (Chow 1.20)

   a. Find constants a, b and c such the the vector field

   $$\mathbf{A} := (x + 2y + az)\,\hat{\mathbf{x}} + (bx - 3y - z)\,\hat{\mathbf{y}} + (4x + cy + 2z)\,\hat{\mathbf{z}}$$

   is irrotational (i.e., curl-free).

   b. Show that the resulting vector field can be expressed as the gradient of a scalar field.