Physics 5115 — Florida Atlantic University
Mathematical Physics — Fall 2010

Problem Set I

Due: Tuesday, 7 September 2010

- Do (at least) **four** of the following five problems from the text.
- Solutions are due (no later than) at the **beginning** of class.

1. (Exercise A.2, p. 747)
   Let U, V and W be vector spaces, and let $A : V \to W$ and $B : U \to V$ be linear maps. Suppose that these maps are represented by matrices $\boldsymbol{A}$, with entries $A^{\mu}{}_{\nu}$, and $\boldsymbol{B}$, with entries $B^{\mu}{}_{\nu}$, respectively, relative to given bases on the three vector spaces. Use the action of the maps on basis elements to show that the product map $A \circ B : U \to W$ is represented by the matrix product $\boldsymbol{AB}$, with entries $A^{\mu}{}_{\lambda}\, B^{\lambda}{}_{\nu}$, relative to the given bases.

2. (Exercises A.3, A.4 and A.5, p. 753)
   Let $A : V \to V$ and $B : V \to V$ be linear operators mapping a vector space V to itself.
   a. Show that $(AB)^* = B^* A^*$.
   b. How does the reversal of operator order in $(AB)^* = B^* A^*$ manifest itself in the Dirac notation?
   c. Suppose that the operator A is represented by the matrix $\boldsymbol{A}$ relative to a given basis $\mathbf{e}_{\mu}$ on V. Show that the conjugate operator A^* is represented by the transpose $\boldsymbol{A}^{\top}$ of the matrix $\boldsymbol{A}$ relative the dual basis $\mathbf{e}^{*\mu}$ on the dual space V^*.

3. (Exercise A.8, p. 756)
   Let P_1 be a projection operator. Show that
   a. $P_2 := I - P_1$ is also a projection operator.
   b. $P_1\, P_2 = 0$.
   c. $\mathrm{Im}\, P_2 = \mathrm{Ker}\, P_1$ and $\mathrm{Ker}\, P_2 = \mathrm{Im}\, P_1$.

4. (Exercise A.9, p. 762)
   Let ω be a skew-symmetric n-linear form on an n-dimensional vector space V. Assuming that ω does not vanish identically, show that a set $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n\}$ of vectors is linearly independent, and hence forms a basis, if and only if $\omega(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n) \neq 0$.

5. (Exercise A.15, p. 770)
Suppose that V is a vector space of dimension $n > 1$ and that $T : V \to V$ is a linear operator that obeys the equation

$$(T - \lambda I)^p = 0$$

for $p = n$, but not for any smaller p. Here, λ is a scalar and I is the identity operator.

a. Show that every eigenvector of T must have eigenvalue λ. Use this result to deduce that T cannot be diagonalized.

b. Show that there must exist a vector $\mathbf{e}_1$ such that $(T - \lambda I)^p\, \mathbf{e}_1 = \mathbf{0}$ for $p = n$, but not for any smaller value of p.

c. Define the vectors $\mathbf{e}_2 := (T - \lambda I)\, \mathbf{e}_1$, $\mathbf{e}_3 := (T - \lambda I)\, \mathbf{e}_2$, and so forth up to $\mathbf{e}_n$. Show that these vectors must be linearly independent, and therefore form a basis for V.

d. Write out the $n \times n$ matrix $\boldsymbol{T}$ representing the operator T in the basis $\{\mathbf{e}_1, \cdots \mathbf{e}_n\}$.