Physics 5115 Florida Atlantic University
Mathematical Physics Fall 2010

Problem Set II

Due: Thursday, 16 September 2010

- Do (at least) **four** of the following five problems from the text.
- Solutions are due (no later than) at the **beginning** of class.

1. (Exercise 1.2, p. 38)

   Fermat's principle states that a ray of light follows the path of least time between two given points $\mathbf{a}$ and $\mathbf{b}$ in an optical medium with refractive index $n(\mathbf{x})$. The refractive index is defined such that the speed of light is $c/n(\mathbf{x})$ at each point of the medium. Assume that the ray in question travels in the xy-plane through a layered medium with variable $n(\mathbf{x})$.

   a. Suppose that $n(x)$ depends only on the coordinate x. Use Fermat's principle to establish Snell's law in its general form $n(x)\sin\psi = \text{const.}$ by finding the stationary paths of

   $$F_x[y] := \int_{x_a}^{x_b} n(x)\sqrt{1+y'^2(x)}\,\mathrm{d}x,$$

   where as usual the prime denotes differentiation with respect to x.

   b. Now suppose that $n(y)$ depends only on the coordinate y and derive Snell's law by finding the stationary paths of

   $$F_y[y] := \int_{x_a}^{x_b} n\big(y(x)\big)\sqrt{1+y'^2(x)}\,\mathrm{d}x.$$

   Show that your result in this case is physically equivalent to that of part (a).

   Hint: You may find it easiest to use the first integral of Euler's equation.

2. (Exercise 1.8, p. 43)

   The Lagrangian for a particle of mass m and charge q is

   $$L(\mathbf{x}, \dot{\mathbf{x}}) = \frac{m\,\dot{\mathbf{x}}^2}{2} - q\,\phi(\mathbf{x}) + q\,\dot{\mathbf{x}}\cdot\mathbf{A}(\mathbf{x}).$$

   Show that Lagrange's equation leads to

   $$m\ddot{\mathbf{x}} = q\,\mathbf{E}(\mathbf{x}) + \dot{\mathbf{x}}\times\mathbf{B}(\mathbf{x}) \qquad \text{with} \qquad \mathbf{E}(\mathbf{x}) = -\boldsymbol{\nabla}\phi(\mathbf{x}) - \frac{\partial\mathbf{A}}{\partial t}(\mathbf{x}) \quad \text{and} \quad \mathbf{B}(\mathbf{x}) := \boldsymbol{\nabla}\times\mathbf{A}(\mathbf{x}).$$

3. (Exercise 1.13, p. 46)

   The shape of a distorted drumskin is described by the function $h(x,y)$, which gives the height to which the point (x,y) of the flat, undistorted drumskin is displaced.

   a. Show that the area of the distorted drumskin is equal to

   $$A[h] := \int \sqrt{1 + h_{,x}^2(x,y) + h_{,y}^2(x,y)}\,\mathrm{d}x\,\mathrm{d}y,$$

   where $h_{,x}(x,y)$ denotes the partial derivative of $h(x,y)$ with respect to x at the point (x,y) and the integral is taken over the area of the flat drumskin.

   b. For small distortions, show that the area reduces to

   $$a[h] := A_0 + \frac{1}{2}\int \left|\boldsymbol{\nabla} h(x,y)\right|^2 \mathrm{d}x\,\mathrm{d}y,$$

   where A_0 is the total area of the undistorted drumskin.

   c. Show that the stationary points of $a[h]$ under variations that vanish at the boundary are solutions $h(x,y)$ of the two-dimensional Laplace equation.

   d. Suppose that the drumskin has uniform mass per unit area ρ_0 and surface tension T_0. Write down the Lagrangian controling the motion of the drumskin (in the limit of small distortions) and derive the equation of motion that follows from it.

4. (Problem 1.6, p. 41)

   We may describe a catenary curve in parametric form $x = x(s)$, $y = y(s)$, where s is the arc-length along the curve. The potential energy is then simply

   $$E[x,y] := \int_0^L \rho g y(s)\,\mathrm{d}s,$$

   where ρ is the mass per unit length of the hanging chain. However, $x(s)$ and $y(s)$ cannot be independent functions of s in this formulation because $\dot{x}^2(s) + \dot{y}^2(s) = 1$ at every point of the curve. The dots here denote derivatives with respect to s.

   a. Introduce a continuous family $\lambda(s)$ of Lagrange multipliers to enforce the constraint on $\dot{x}(s)$ and $\dot{y}(s)$ at each point of the curve. Derive coupled equations for $x(s)$ and $y(s)$ from the resulting functional. By thinking about the forces acting on a short length of chain, show that $\lambda(s)$ is proportional to the tension $T(s)$ in the chain at each point.

   Hint: You may find it helpful to introduce the angle $\psi(s)$ with $\dot{x}(s) = \cos\psi(s)$ and $\dot{y}(s) = -\sin\psi(s)$, so that s and ψ define *intrinsic coordinates* for the curve.

   b. You are provided with a lightweight line of length $\pi a/2$ and some lead shot of total mass M. Modify your equations from the previous part to allow for a variable mass per unit length $\rho(s)$, which we can achieve by attaching variable amounts of shot to each point along the line. Determine how the lead should be distributed if the loaded line is to hang in a quarter-circular arc of radius a (see Figure 1.15 on p. 42 of the text) when its ends are attached to two points at the same height.

5. (Problem 1.12, p. 45)
   Consider the action functional

$$S[\mathbf{v},\rho,\phi,\beta,\gamma] := \iint \Big\{ -\tfrac{1}{2}\,\rho\,\mathbf{v}^2 - \phi\left[\dot{\rho} + \boldsymbol{\nabla}\cdot(\rho\,\mathbf{v})\right] + \rho\,\beta\left[\dot{\gamma} + (\mathbf{v}\cdot\boldsymbol{\nabla})\gamma\right] + u(\rho) \Big\}\, \mathrm{d}^3x\,\mathrm{d}t,$$

   which generalizes Eq. (1.177) from the text to include two new scalar fields β and γ. All of the fields on which this action depends are functions both of position $\mathbf{x}$ and time t, and dots denote partial derivatives with respect to time. As before, the function $u(\rho)$ describes the internal energy density at given mass density.

   a. Show that varying the velocity field $\mathbf{v}$ leads to

$$\mathbf{v} = \boldsymbol{\nabla}\phi + \beta\,\boldsymbol{\nabla}\gamma \qquad \rightsquigarrow \qquad \boldsymbol{\omega} := \boldsymbol{\nabla}\times\mathbf{v} = \boldsymbol{\nabla}\beta\times\boldsymbol{\nabla}\gamma.$$

   This is called the ***Clebsch representation*** of the velocity field, which allows for non-zero vorticity $\boldsymbol{\omega}$ of the fluid flow.

   b. Show that varying the remaining fields ρ, ϕ, β and γ collectively imply

$$\dot{\rho} = \boldsymbol{\nabla}\cdot(\rho\,\mathbf{v}) \qquad \text{and} \qquad \dot{\mathbf{v}} + \boldsymbol{\omega}\times\mathbf{v} = -\boldsymbol{\nabla}\left[\tfrac{1}{2}\,\mathbf{v}^2 + u'(\rho)\right],$$

   where the prime denotes the derivative of u with respect to its argument ρ. Physically, these are the mass conservation equation and Bernoulli's equation for this problem.

   c. Show that the above form of Bernouli's equation is equivalent to Euler's equation

$$\dot{\mathbf{v}} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = -\boldsymbol{\nabla}u'(\rho).$$

   Consequently S provides an action principle for a general inviscid barotropic flow.