Physics 5115 Florida Atlantic University
Mathematical Physics Fall 2010

Problem Set III

Due: Tuesday, 28 September 2010

- Do (at least) **four** of the following five problems from the text.
- Solutions are due (no later than) at the **beginning** of class.

1. (Exercise B.1, p. 790)

   Suppose that we exponentially suppress high frequencies by multiplying the Fourier amplitude $\tilde{f}(k)$ by $\mathrm{e}^{-\epsilon|k|}$. Show that the original signal $f(x)$ is smoothed by convolution with a ***Lorentzian approximation*** to the delta function

$$\delta_\epsilon^{\mathrm{L}}(x-\xi) = \frac{1}{\pi}\frac{\epsilon}{\epsilon^2 + (x-\xi)^2}.$$

   As $\epsilon \to 0$, observe that $\delta_\epsilon^{\mathrm{L}}(x) \to \delta(x)$ in the sense of distributions.

2. (Exercises B.3, p. 790, and B.6, p. 792, *the Hilbert transform*)

   a. Show that the sum

$$D_r(\theta) := \sum_{n=-\infty}^{\infty} \operatorname{sgn}(n)\,\mathrm{e}^{\mathrm{i}n\theta}\, r^{|n|} = \frac{r\,\mathrm{e}^{\mathrm{i}\theta}}{1 - r\,\mathrm{e}^{\mathrm{i}\theta}} - \frac{r\,\mathrm{e}^{-\mathrm{i}\theta}}{1 - r\,\mathrm{e}^{-\mathrm{i}\theta}},$$

   which converges for $0 < r < 1$, approaches the principal-value distribution

$$D(\theta) := \mathrm{i}\,\mathcal{P}\cot\frac{\theta}{2}$$

   in the limit $r \to 1$.

   b. Let $f(\theta)$ be a smooth function on the unit circle and define its ***Hilbert transform***

$$\mathcal{H}f(\theta) := \frac{1}{2\pi}\,⨍_{-\pi}^{\pi} f(\theta')\cot\left(\frac{\theta-\theta'}{2}\right)\mathrm{d}\theta'.$$

   Show that $f(\theta)$ can be recovered if one knows both its Hilbert transform $\mathcal{H}f(\theta)$ and its average value $\langle f\rangle$, according to the formula

$$f(\theta) = -\frac{1}{2\pi}\,⨍_{-\pi}^{\pi} \mathcal{H}f(\theta')\cot\left(\frac{\theta-\theta'}{2}\right)\mathrm{d}\theta' + \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta')\,\mathrm{d}\theta' =: -\mathcal{H}^2 f(\theta) + \langle f\rangle.$$

   c. Let $f(x)$ be a function on the real line such that $\int_{-\infty}^{\infty} |f(x)|\,\mathrm{d}x$ is finite. Take a suitable limit in the previous result to show that $\mathcal{H}^2 f(x) = -f(x)$, where

$$\mathcal{H}f(x) := \frac{1}{\pi}\,⨍_{-\infty}^{\infty} \frac{f(x')}{x-x'}\,\mathrm{d}x'$$

   defines the Hilbert transform of a function on the real line.

3. (Exercises 2.3, p. 64, and 2.5, p. 65)

   a. Evaluate the integral
   $$F(s,t) = \int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\,\mathrm{e}^{2sx-s^2}\,\mathrm{e}^{2tx-t^2}\,\mathrm{d}x$$
   and expand both sides of your result as double power series in s and t. By comparing the coefficients of $s^m\,t^n$ on either side, show that
   $$\int_{-\infty}^{\infty} H_m(x)\,H_n(x)\,\mathrm{e}^{-x^2}\,\mathrm{d}x = 2^n\,n!\,\sqrt{\pi}\,\delta_{mn}.$$

   b. Define the ***normalized Hermite functions***
   $$\varphi_n(x) := \frac{1}{\sqrt{2^n\,n!\,\sqrt{\pi}}}\,H_n(x)\,\mathrm{e}^{-x^2/2}$$
   and the ***Fourier transform operator***
   $$\mathcal{F}f(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}xs}\,f(s)\,\mathrm{d}s.$$
   Note that $\mathcal{F}^4$ is the identity map when the integral is normalized in this way, whence the only possible eigenvalues of $\mathcal{F}$ are ± 1 and $\pm\mathrm{i}$. Starting from Eq. (2.56) of the book, or otherwise, show that $\varphi_n(x)$ is an eigenfunction of $\mathcal{F}$ with eigenvalue i^n.

4. (Exercise 2.13, p. 78)
   The completeness of a set $\{P_n(x)\}$ of polynomials that are orthonormal with respect to a positive weight function $w(x)$ may be expressed mathematically in the form
   $$\sum_{n=0}^{\infty} P_n(x)\,P_n(y) = \frac{\delta(x-y)}{w(x)}.$$
   It is sometimes useful to have a formula for the partial sums of this infinite series. Suppose that the $P_n(x)$ obey the three-term recurrence relation
   $$x\,P_n(x) = b_n\,P_{n+1}(x) + a_n\,P_n(x) + b_{n-1}\,P_{n-1}(x),$$
   subject to the initial conditions
   $$P_{-1}(x) = 0 \qquad \text{and} \qquad P_0(x) = 1.$$
   Use this recurrence relation, together with its initial conditions, to obtain the ***Christoffel–Darboux partial sum*** formula
   $$\sum_{n=0}^{N-1} P_n(x)\,P_n(y) = b_{N-1}\,\frac{P_N(x)\,P_{N-1}(y) - P_{N-1}(x)\,P_N(y)}{x-y}.$$

5. (Exercises 2.20, 2.21 and 2.22, p. 84)

a. Let $f(x)$ be a continuous function. Observe that $f(x)\,\delta(x) = f(0)\,\delta(x)$ to deduce that

$$\frac{\mathrm{d}}{\mathrm{d}x}\big[f(x)\,\delta(x)\big] = f(0)\,\delta'(x).$$

If $f(x)$ is not only continuous but differentiable , then we can use the product rule to compute the above derivative in the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\big[f(x)\,\delta(x)\big] = f'(x)\,\delta(x) + f(x)\,\delta'(x).$$

Show that these two expressions are equivalent in the sense of distributions by integrating the right side of each against an arbitrary test function $\varphi(x)$.

b. Let $\varphi(x)$ be a test function. Show that

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\frac{\mathcal{P}}{x-t}\,\varphi(x)\,\mathrm{d}x = \fint_{-\infty}^{\infty}\frac{\varphi(x)-\varphi(t)}{(x-t)^2}\,\mathrm{d}x.$$

Show further that the right-hand side of this equation is equal to

$$-\left(\frac{\partial}{\partial x}\,\frac{\mathcal{P}}{x-t},\varphi\right) := \int_{-\infty}^{\infty}\frac{\mathcal{P}}{x-t}\,\varphi'(x)\,\mathrm{d}x.$$

c. Let $\theta(x)$ denote the ***step function*** or ***Heaviside distribution***

$$\theta(x) := \begin{cases} 1 & x > 0 \\ \text{undefined} & x = 0 \\ 0 & x < 0. \end{cases}$$

Derive the equation

$$\lim_{\epsilon\to 0^+}\ln(x+\mathrm{i}\epsilon) = \ln|x| + \mathrm{i}\pi\,\theta(-x),$$

and take the weak derivative of both sides to show that

$$\lim_{\epsilon\to 0^+}\frac{1}{x+\mathrm{i}\epsilon} = \frac{\mathcal{P}}{x} - \mathrm{i}\pi\,\delta(x).$$