A <u>vector space</u> is a set in which one can form <u>linear combinations</u>:

$$\left.\begin{array}{l} v_1, v_2 \in V \\ \alpha_1, \alpha_2 \in \mathbb{F} \end{array}\right\} \rightsquigarrow \alpha_1 v_1 + \alpha_2 v_2 \in V$$

(subject to natural axioms.)

Example: Bessel Functions

$$0 = z^2 B''(z) + z B'(z) + (z^2 - \nu^2) B(z)$$

$$\Rightarrow \quad \begin{array}{l} B(z) = B_1 J_\nu(z) + B_2 N_\nu(z) \\ B(z) = B_+ H_\nu^+(z) + B_- H_\nu^-(z) \end{array}$$

$$\rightsquigarrow \quad B(z) \longleftrightarrow \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \longleftrightarrow \begin{pmatrix} B_+ \\ B_- \end{pmatrix}$$

The set of Bessel functions of order ν is naturally an <u>abstract</u> vector space.

A subset $B \subset V$ is called a basis if it is

(a) linearly independent:

$$\alpha^1 b_1 + \cdots + \alpha^n b_n = 0$$

$$\Rightarrow \alpha^1 = \alpha^2 = \cdots = \alpha^n = 0$$

(b) spans V:

every $v \in V$ can be written as a finite linear combination of the vectors in B.

Thm: If V admits a finite basis B, then any other basis B' contains the same number of vectors as B.

Example: $\{J_\nu(z), N_\nu(z)\}$ and $\{H^\pm_\nu(z)\}$ are both bases for the Bessel functions of order ν.

The number of elements in a basis B is called the <u>dimension</u> of a vector space V.

Change of Basis

Let B and $\tilde{B}$ be two bases for a vector space V.

$$\Rightarrow \tilde{b}_\beta = \sum_\alpha \lambda^\alpha{}_\beta \, b_\alpha \leftarrow B = \text{basis}$$

$$b_\alpha = \sum_\beta \tilde{\lambda}^\beta{}_\alpha \, \tilde{b}_\beta \leftarrow \tilde{B} = \text{basis}$$

We can write this in matrix form:

$$(\tilde{b}_1 \cdots \tilde{b}_n) = (b_1 \cdots b_n) \underbrace{\begin{pmatrix} \lambda^1{}_1 & \cdots & \lambda^1{}_n \\ \vdots & \ddots & \vdots \\ \lambda^n{}_1 & \cdots & \lambda^n{}_n \end{pmatrix}}_{\Lambda \,=\, \text{change-of-basis matrix}}$$

Example:

$$\begin{pmatrix} H^+_\nu & H^-_\nu \end{pmatrix} = \begin{pmatrix} J_\nu & N_\nu \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

$$\begin{pmatrix} J_\nu & N_\nu \end{pmatrix} = \begin{pmatrix} H^+_\nu & H^-_\nu \end{pmatrix} \begin{pmatrix} 1/2 & -i/2 \\ 1/2 & i/2 \end{pmatrix}$$

Note: $\tilde{\Lambda} = \Lambda^{-1}$ (why?)

A map $\varphi : V \to W$ is called <u>linear</u> if it commutes with the formation of linear combinations:

$$\varphi(\underbrace{\alpha_1 v_1 + \alpha_2 v_2}_{\text{sum in } V}) = \underbrace{\alpha_1 \varphi(v_1) + \alpha_2 \varphi(v_2)}_{\text{sum in } W}$$

If we choose bases B on V and C on W, we can write

$$\begin{aligned}
\varphi(v) &= \varphi\Big(\sum_\alpha v^\alpha b_\alpha\Big) \\
&= \sum_\alpha v^\alpha \varphi(b_\alpha) \\
&= \sum_\alpha v^\alpha \Big[\sum_\beta \varphi^\beta{}_\alpha c_\beta\Big] \\
&= \sum_{\alpha\beta} \varphi^\beta{}_\alpha v^\alpha c_\beta \\
&= (c_1 \cdots c_m) \begin{pmatrix} \varphi^1{}_1 & \cdots & \varphi^1{}_n \\ \vdots & \ddots & \vdots \\ \varphi^m{}_1 & \cdots & \varphi^m{}_n \end{pmatrix} \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix}
\end{aligned}$$

expansion of $\varphi(b_\alpha) \in W$

matrix representation

Dual Basis

V, basis $B = \{ b_\alpha \}$

$$\eta^\alpha(v) = v^\alpha$$

given $v = \sum_\alpha v^\alpha b_\alpha$

Given bases B for V and C for W, we naturally get a basis for vector space $\mathrm{Hom}(V, W)$ of linear $\varphi : V \to W$

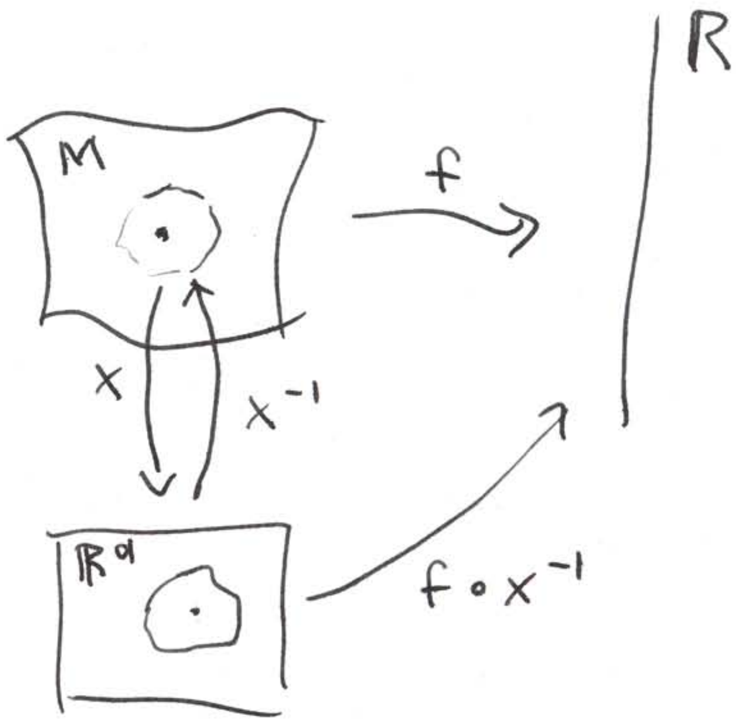
$$\varphi(v) = \sum_{\alpha \beta} \varphi^\beta{}_\alpha \, v^\alpha \, c_\beta$$

$$\eta^\alpha(v)$$

$$\sum_{\alpha \beta} \varphi^\beta{}_\alpha (c_\beta \otimes \eta^\alpha)(v) = \sum_{\alpha \beta} c_\beta \, \varphi^\beta{}_\alpha \, \eta^\alpha(v)$$

Manifolds

A <u>differentiable manifold</u> is a set on which one can identify the <u>smooth functions</u>.



Adjoint Map

Suppose $\varphi : V \to W$ is linear.

Define $\varphi^* : W^* \to V^*$

$$\varphi^*(\zeta)(v) := \zeta(\varphi(v))$$

$\in W^*$

$$R_{\alpha\beta\gamma}{}^{\delta} \in V^* \otimes V^* \otimes V^* \otimes V$$

$$R(X, Y, Z) = W$$

$\varphi \in W \otimes V^*$

$\wr$

linear maps $\quad V \to W \quad \varphi^{\alpha}{}_{\beta} v^{\beta}$

$W^* \to V^* \quad \varphi^{\alpha}{}_{\beta} \zeta_{\alpha}$

$R_{abc}{}^{d} \quad W^* \otimes V \to \mathbb{F} \quad \varphi^{\alpha}{}_{\beta} \psi^{\beta}{}_{\alpha}$

$\mathbb{F} \to W \otimes V^* \quad c\varphi^{\alpha}{}_{\beta}$

$$\varphi(v) = \left[\sum_{\alpha\beta} \varphi^{\alpha}{}_{\beta}\, c_\alpha \otimes \eta^{\beta}\right](v)$$

$$\left[c_\alpha \otimes \eta^{\beta}\right](v) := \eta^{\beta}(v) \cdot c_\alpha$$

Space spanned by $c_\alpha \otimes \eta^{\beta}$

is vector space denoted

$$W \otimes V^*$$

$$c_\alpha \quad \eta^{\beta}$$

Riemann tensor $\in V^* \otimes V^* \otimes V^* \otimes V$

$$R_{\alpha\beta\sigma}{}^{\delta}$$

Space of linear maps $V \to W$ is itself a vector space.

$$(\alpha_1 \varphi_1 + \alpha_2 \varphi_2)(v)$$

$$:= \alpha_1 \varphi_1(v) + \alpha_2 \varphi_2(v)$$

$\rightsquigarrow$ Dual Space V^*

= linear maps $\omega : V \to \mathbb{F}$

$$\omega(v_1) = 0 = \omega(v_2)$$

$$\omega(\alpha_1 v_1 + \alpha_2 v_2)$$

$$= \alpha_1 \omega(v_1) + \alpha_2 \omega(v_2) = 0$$

$\cancel{A_\omega = \{\omega}$ $\quad A_\omega := \{ v \mid \omega(v) = 1 \}$

$$A_{2\omega} := \{ v \mid 2\omega(v) = 1 \}$$

$$= \{ v \mid \omega(v) = \tfrac{1}{2} \}$$