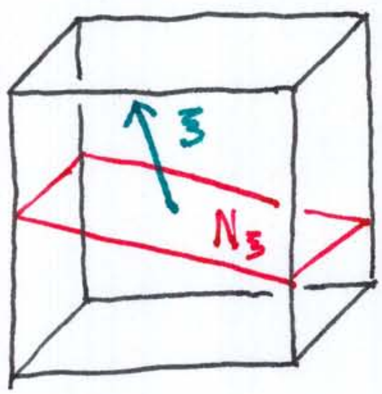
Lecture 16

The Schwarzschild Metric

What is the real geometry of spacetime near the sun?

Hypersurface Orthogonality

Any vector ξ^a in a space of dimension n with metric g_{ab} defines an $(n-1)$-dimensional orthogonal subspace N_ξ



N_ξ is the set $\{ V \mid \xi \cdot V = 0 \}$
It is the subspace associated with

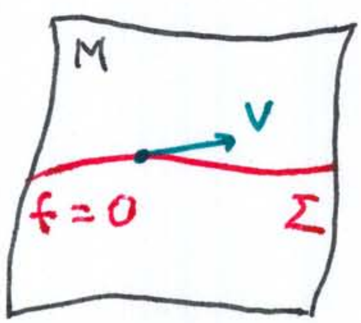
$$\xi_a := g_{ab}\, \xi^b$$

A vector field on a manifold M with metric defines a distribution of such subspaces in the tangent spaces at each point.

These subspaces can <u>twist</u> from point to point such that there is no <u>submanifold</u> Σ of M whose tangent spaces are those in the distribution.

Idea: If Σ exists, then there is a function f with $f = 0$ on Σ and $df \neq 0$ on Σ

$$V^a \nabla_a f = 0 \quad \Rightarrow \quad V \parallel \Sigma$$



$$\Rightarrow \quad V \cdot \xi = 0$$

$$\therefore \quad \xi_a \propto \nabla_a f$$

$$\xi_a = g \nabla_a f$$

In differential forms notation:

$$\xi = g\, df \quad \Rightarrow \quad d\xi = dg \wedge df$$

$$\xi \wedge d\xi = 0 \quad \Leftarrow \qquad\qquad = \frac{dg}{g} \wedge \xi$$

Examples in Minkowski:

$$ds^2 = -dt^2 + dx^2 + dy^2$$

1) $\xi = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x}$

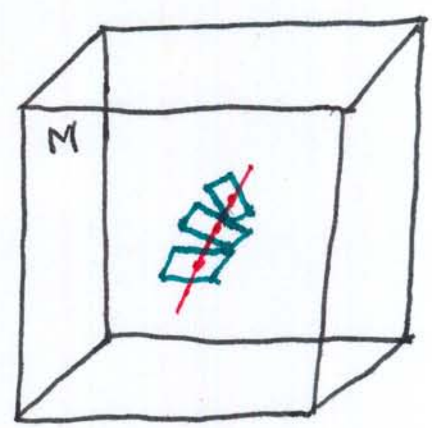
$$\Rightarrow \quad g(\xi) = -dt + u\,dx$$

$$d\xi = 0 + du \wedge dx$$

$$\xi \wedge d\xi = -dt \wedge du \wedge dx$$

$$= \frac{\partial u}{\partial y}\, dt \wedge dx \wedge dy$$

So, ξ is hypersurface-orthogonal if and only if $\frac{\partial u}{\partial y} = 0$.

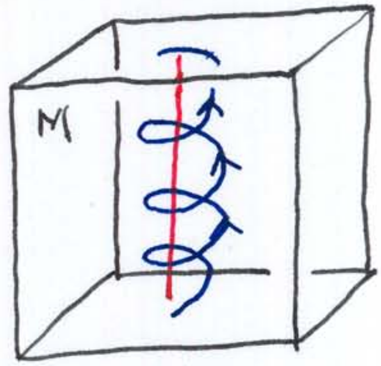


$\frac{\partial}{\partial y}$ always $\perp$

$u\frac{\partial}{\partial t} + \frac{\partial}{\partial x}$ can

<u>twist</u>

2) $\xi = \frac{\partial}{\partial t} + \Omega x \frac{\partial}{\partial y} - \Omega y \frac{\partial}{\partial x}$



$$g(\xi) = -dt + \Omega x\, dy - \Omega y\, dx$$

$$d\xi = 0 + \Omega\, dx \wedge dy - \Omega\, dy \wedge dx$$

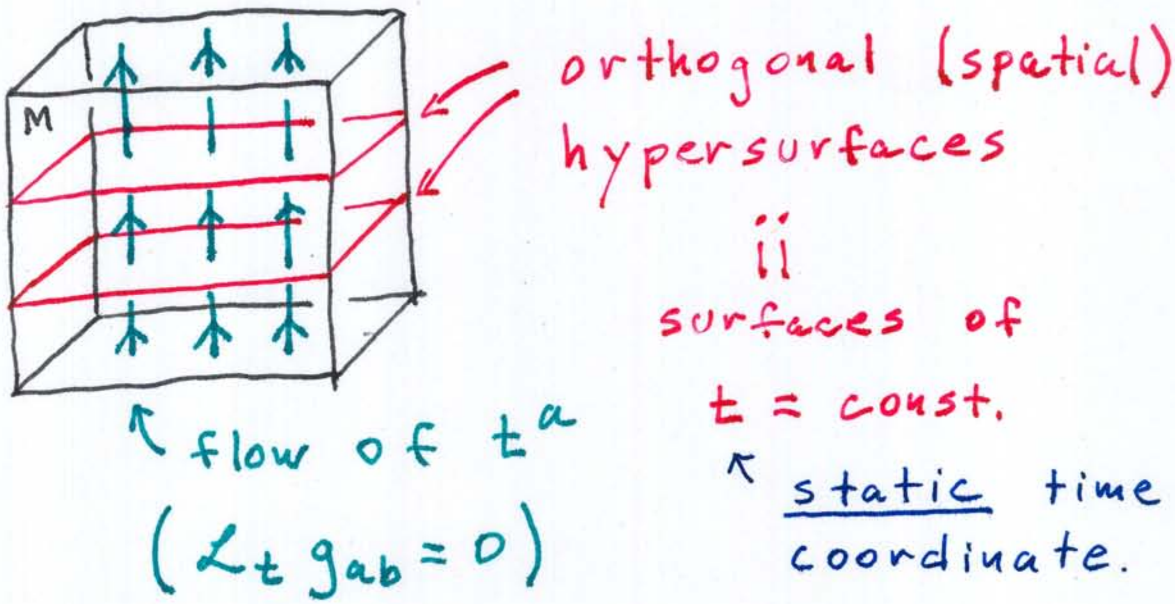
$$= 2\Omega\, dx \wedge dy$$

$$\xi \wedge d\xi = -2\Omega\, dt \wedge dx \wedge dy$$

<u>Physically</u>, twisting vector fields are often associated with <u>rotation</u>.

Static Spacetimes

A spacetime (M, g_{ab}) is said to be <u>static</u> if it admits a <u>hypersurface-orthogonal time-like</u> Killing field t^a.



<u>Define</u> the time coordinate t to be constant on each orthogonal hypersurface and

$$t^a \nabla_a t = 1 \leftarrow \text{normalization.}$$

Spherical Symmetry

A spacetime (M, g_{ab}) is said to be spherically symmetric if the Lie algebra of its Killing fields contains $su(2)$ as a sub-algebra:

L_i^{a} Killing fields $i = 1, 2, 3$

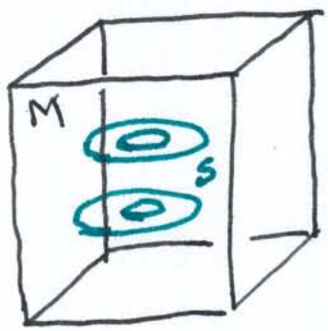
$$[L_i, L_j]^a = \varepsilon_{ij}{}^{k} L_k^a$$

and

at every point these Killing fields are tangent to a two-dimensional sphere.

(Second condition is integrability. It is not guaranteed.

So, a spherically symmetric spacetime is foliated by round spheres S:



Each S has an area A_S.

$$r_S := \sqrt{\frac{A_S}{4\pi}}$$

Thus, spherical symmetry gives us an areal radius coordinate:

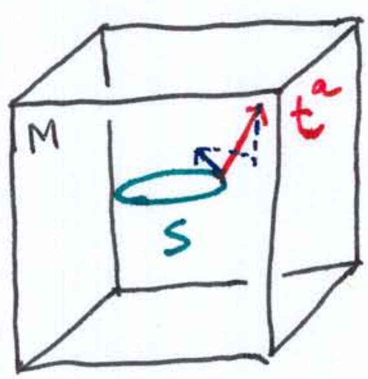
$$p \rightsquigarrow S \text{ containing } p$$
$$\rightsquigarrow r_S =: r(p) \leftarrow \text{function on } M$$

In addition, the intrinsic (two-dimensional) metric on each S is

$$^2ds^2 = r_S^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

Static Spherical Spacetimes

When spacetime is <u>both</u> static <u>and</u> spherically symmetric, the Killing field t^a must be orthogonal to the spheres S



If the projection of t^a into S is non-zero, that projection <u>breaks</u> spherical symmetry in <u>spacetime</u>.

Thus, the symmetry spheres lie <u>within</u> the static slices.

The metric on a static spacetime has the form

$$ds^2 = -N^2(\vec{x})\,dt^2 + h_{ij}(\vec{x})\,dx^i dx^j$$

If that spacetime is <u>also</u> spherically symmetric, we can simplify further:

$$ds^2 = -N^2(r)\,dt^2 + f^2(r)\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

That is, we have a complete set of <u>natural</u> coordinates (t, r, θ, ϕ) on spacetime, and the metric is given by <u>two</u> functions (N, f) of <u>one</u> of those coordinates (r).

Cartan's Method

To calculate the curvature of a static spherically symmetric metric, we use a method due to Cartan based on an orthonormal basis of fields:

$$g_{ab}\, e^a_\alpha\, e^b_\beta = \eta_{\alpha\beta} = \pm 1, 0$$

$$\rightsquigarrow g_{ab} = \eta_{\alpha\beta}\, \underbrace{e^\alpha_a\, e^\beta_b}_{\text{dual basis fields}}$$

Let ∇_a denote the curved torsion-free metric connection, and D_a the flat torsive basis connection.

$$\nabla_a g_{bc} = 0$$

$$D_a e^\beta_b = 0$$

We have

$$\nabla_a e_b^{\beta} = D_a e_b^{\beta} + C_{ab}{}^c e_c^{\beta}$$

$$= C_{ab}{}^c e_c^{\beta}$$

by definition of D_a, and

$$\nabla_a g_{bc} = D_a g_{bc} + C_{ab}{}^m g_{mc} + C_{ac}{}^m g_{bm}$$

$$\Rightarrow \quad 0 = D_a (\eta_{\beta\gamma} e_b^{\beta} e_c^{\gamma}) + 2 C_{a(bc)}$$

$$= e_b^{\gamma} e_c^{\gamma} \cancelto{0}{D_a \eta_{\beta\gamma}} + 2 C_{a(bc)}$$

(<u>orthonormal</u> basis)

Let $T_{ab}{}^c$ denote the torsion of D_a and we have

$$2 \nabla_{[a} \nabla_{b]} f = 2 D_{[a} D_{b]} f + 2 C_{[ab]}{}^c D_c f$$

$$\Rightarrow \quad 0 = - T_{ab}{}^c D_c f + 2 C_{[ab]}{}^c D_c f$$

$$\Rightarrow 2 C_{[ab]c} = T_{abc}, \quad C_{a(bc)} = 0$$

We can solve these for C_{abc} in the usual way:

$$C_{abc} = T_{abc} + C_{bac}$$

$$= T_{abc} - C_{bca} \leftarrow \text{cyclic permutation}$$

$$= T_{abc} - (T_{bca} - C_{cab})$$

$$= T_{abc} - T_{bca} + T_{cab} - C_{abc}$$

$$\Rightarrow 2C_{abc} = 2T_{c(ab)} + T_{abc}$$

$$= 2T_{a[bc]} - T_{bca}$$

This is fine, but Cartan has a clever way to <u>organize</u> the calculation using differential forms.

Recall that

$$2\nabla_{[a}\alpha_{b]} = 2\partial_{[a}\alpha_{b]} + 2\Gamma_{[ab]}{}^{c}\alpha_c$$
$$= (d\alpha)_{ab} - T_{ab}{}^{c}\alpha_c$$

So, in the present case

$$2\nabla_{[a}\alpha_{\beta]} = (d\alpha)_{ab}$$

$$2D_{[a}\alpha_{\beta]} = (d\alpha)_{ab} - T_{ab}{}^{c}\alpha_c$$

Take $\alpha_b = e_b^{\beta}$ and we find

$$2\nabla_{[a}e_{b]}^{\beta} = \cancelto{0}{2D_{[a}e_{b]}^{\beta}} + 2C_{[ab]}{}^{c}e_c^{\beta}$$
$$= 2\delta_{[b}^{m}C_{a]m}{}^{c}e_c^{\beta}$$
$$= 2e_{[b}^{\alpha}\left(C_{a]m}{}^{c}e_{\alpha}^{m}e_c^{\beta}\right)$$

<u>matrix</u> of 1-forms $\rightarrow \omega_{a]\alpha}{}^{\beta}$

$$de^{\beta} = \cancel{+}\cancel{2}\,\omega_{\alpha}{}^{\beta}\wedge e^{\alpha}$$

But remember that $C_{a(bc)}=0$ which implies that

$$\omega_{a(\beta\gamma)} = \omega_{a(\beta}{}^{\mu}\,\eta_{\gamma)\mu}$$

$$= C_{ab}{}^{c}\, e_c^{\mu}\, e^b_{(\beta}\,\eta_{\gamma)\mu}$$

$$= C_{abc}\, e^c_{\mu}\, e^b_{(\beta}\,\delta^{\mu}_{\gamma)}$$

$$= C_{a(bc)}\, e^b_{(\beta}\, e^c_{\gamma)} = 0$$

Thus, we find that

$$\eta_{\alpha\beta}\, de^{\beta} = \eta_{\alpha\beta}\, \omega_{\gamma}{}^{\beta} \wedge e^{\gamma}$$

$$= -\omega_{\alpha\beta} \wedge e^{\cancel{\gamma}\,\beta}$$

<u>anti-symmetric</u> matrix

of 1-forms

These are the <u>Cartan</u> <u>connection forms</u>.

Meanwhile, we have

$$
\begin{aligned}
R_{abc}{}^{d} &= 2 D_{[a} C_{b]c}{}^{d} + T_{ab}{}^{m} C_{mc}{}^{d} \\
&\qquad + 2 C_{[a|c|}{}^{m} C_{b]m}{}^{d} \\
&= 2 D_{[a} \left(\omega_{b]\alpha}{}^{\beta} e^{\alpha}_{c} e^{d}_{\beta} \right) \\
&\qquad + T_{ab}{}^{m} \omega_{m\alpha}{}^{\beta} e^{\alpha}_{c} e^{d}_{\beta} \\
&\qquad + 2 \left(e^{\alpha}_{c} e^{m}_{\mu} \omega_{[a|\alpha|}{}^{\mu} \right) \\
&\qquad\quad \times \left(\omega_{b]\nu}{}^{\beta} e^{\nu}_{m} e^{d}_{\beta} \right) \\
&= e^{\alpha}_{c} e^{d}_{\beta} \left(2 D_{[a} \omega_{b]\alpha}{}^{\beta} + T_{ab}{}^{m} \omega_{m\alpha}{}^{\beta} \right) \\
&\qquad + e^{\alpha}_{c} e^{d}_{\beta} \left(2 \omega_{[a|\alpha|}{}^{\mu} \omega_{b]\mu}{}^{\beta} \right) \\
&= e^{\alpha}_{c} e^{d}_{\beta} \left(d\omega_{\alpha}{}^{\beta} + \omega_{\alpha}{}^{\mu} \wedge \omega_{\mu}{}^{\beta} \right)_{ab}
\end{aligned}
$$

<u>matrix</u> of 2-forms.

Note that the concrete index is lifted here.