Lecture 17

Geodesics of the Schwarzschild Geometry

The Field Equations

Calculating the Einstein tensor for static, spherically symmetric spacetimes takes a while.
(See the posted lecture notes.)

We just state the result:

$$G_{ab} = \frac{1}{r^2}\left(r(1-F^2)\right)' e^t_a e^t_b + \frac{1}{r^2}\left(rF^2(\ln N^2)' - (1-F^2)\right) e^r_a e^r_b + \frac{F}{Nr}\left(r(FN')' + (FN)'\right)\left(e^\theta_a e^\theta_b + e^\phi_a e^\phi_b\right)$$

$$ds^2 = -N^2(r)\,dt^2 + F^{-2}(r)\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

$$e^t = N\,dt \qquad e^\theta = r\,d\theta$$

$$e^r = F^{-1}\,dr \qquad e^\phi = r\sin\theta\, d\phi$$

The vacuum Einstein equation

$$G_{ab} = 0$$

gives three equations for the two undetermined functions $N(r)$ and $F(r)$:

① $0 = G_{tt} = \frac{1}{r^2}\left(r(1-F^2)\right)'$

$$\Rightarrow \quad r(1-F^2) = 2M = \text{const.}$$

$$\Rightarrow \quad F^2(r) = 1 - \frac{2M}{r}$$

② $0 = G_{rr} = \frac{1}{r^2}\left(rF^2(\ln N^2)' - (1-F^2)\right)$

$$\Rightarrow \quad (\ln N^2)' = \frac{2M/r^2}{1 - 2M/r}$$

$$\Rightarrow \quad \ln N^2 = \ln\left|1 - \frac{2M}{r}\right| + \text{const.}$$

$$\Rightarrow \quad N^2(r) = C \cdot \left(1 - \frac{2M}{r}\right)$$

Constant may be absorbed into the time coordinate t.

We have now determined the metric completely:

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2$$
$$+r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

But there is still one equation left to solve!

(3) $$0 = G_{\theta\theta} = \frac{F}{Nr}\left(r(FN')' + (FN)'\right)$$

$$\Rightarrow \frac{1}{2} r (N^2)'' + (N^2)' = 0$$

$$= \frac{1}{2} r \left(\frac{2M}{r^2}\right)' + \frac{2M}{r^2}$$

$$= -r\frac{2M}{r^3} + \frac{2M}{r^2} = 0$$

This equation reduces to $0=0$; the equations are internally consistent. This happens because of <u>Bianchi identities</u>.

Geodesics of Schwarzschild

d'Inverno §7.6 establishes the following result:

Let $z(\tau)$ denote a parameterized curve in spacetime. Set

$$2K := g_{ab}(z(\tau))\, \dot{z}^a(\tau)\, \dot{z}^b(\tau)$$

metric at the point $z(\tau)$

Then the geodesic equation in affine parameterization may be written in the Euler-Lagrange form

$$\frac{\partial K}{\partial z^\alpha} - \frac{d}{d\tau}\left(\frac{\partial K}{\partial \dot{z}^\alpha}\right) = 0$$

and $2K = \pm 1, 0$ along the curve.

In Schwarzschild,

$$2K = -\left(1-\frac{2M}{r}\right)\dot{t}^2 + \left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right)$$

As in ordinary, Newtonian orbital mechanics, any cyclic coordinate (which enters the Lagrangian only through its time derivative) has a conserved momentum:

$$-E = \frac{\partial K}{\partial \dot{t}} = -\left(1-\frac{2M}{r}\right)\dot{t}$$

$$L = \frac{\partial K}{\partial \dot{\phi}} = r^2 \sin^2\theta\,\dot{\phi}$$

Physically, these correspond to the "energy" and (z-component of) "angular momentum" of the curve.

- These quantities are conserved <u>precisely</u> because the Schwarzschild metric has Killing vectors (t^a, l_i^a)

$$\nabla_U (U \cdot K) = K \cdot \nabla_U U + U \cdot \nabla_U K$$
$$= U^a U^b \nabla_a K_b = 0$$

- In fact, all three components of "angular momentum" are conserved.

$\Rightarrow$ We may assume, without loss of generality, that the motion occurs in the "equatorial plane" $\theta = \frac{\pi}{2}$.

$$\Rightarrow \dot{\theta} = 0, \quad \sin\theta = 1$$

Substituting these results back into the variational principle,

$$2K = -\left(1-\frac{2M}{r}\right)\frac{E^2}{\left(1-\frac{2M}{r}\right)^2}$$

$$+\left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\frac{L^2}{r^4}$$

$$\Rightarrow \frac{1}{2}E^2 = \frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(1-\frac{2M}{r}\right)\left(\frac{L^2}{r^2}-2K\right)$$

Thus, the radial motion of a geodesic in Schwarzschild is mathematically identical to that of a Newtonian particle in one dimension moving with energy $\frac{1}{2}E^2$ in the potential

$$V_{eff}(r) = \frac{1}{2}\left(1-\frac{2M}{r}\right)\left(\frac{L^2}{r^2}-2K\right)$$

Time-Like Geodesics

Here, we have $2K = -1$, so

$$V_{eff}(r) = \frac{1}{2} - \frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}$$

- The first term is irrelevant.
- The second is Newtonian gravity.
- The third is the ordinary "centrifugal barrier."
- The fourth is new, and brings in relativistic effects.

Circular Orbits

Circular orbits lie at extrema of the effective potential:

$$0 = \frac{\partial V_{eff}}{\partial r} = \frac{M}{r^2} - \frac{L^2}{r^3} + \frac{3ML^2}{r^4}$$

$$\Rightarrow Mr^2 - L^2 r + 3ML^2 = 0$$

$$\Rightarrow r = R_\pm = \frac{L^2 \pm \sqrt{L^4 - 12M^2L^2}}{2M}$$

contrast this with the Newtonian result:

$$0 = \frac{\partial \overset{\circ}{V}_{eff}}{\partial r} = \frac{M}{r^2} - \frac{L^2}{r^3}$$

$$\Rightarrow r = R_0 = \frac{L^2}{M}$$

The phenomenology of circular motion can be quite different in general relativity!

- If $L^2 < 12M^2$, then there are <u>no</u> circular orbits. A test particle always ends up spiraling inward.
- If $L^2 > 12M^2$, then $r = R_+$ is <u>stable</u> (minimum of V_{eff}) but $r = R_-$ is <u>unstable</u>.

Note: if $L^2 = 12M^2$, then $R_+ = R_- = 6M$. This radius defines the <u>innermost stable circular orbit</u> (<u>ISCO</u>) of Schwarzschild.

Note: as $L^2 \to \infty$, $R_+ \to \infty$ but

$$R_- \to \frac{L^2}{2M}\left(1 - \sqrt{1 - \frac{12M^2}{L^2}}\right) \to \frac{L^2}{2M} \cdot \frac{1}{2}\frac{12M^2}{L^2}$$

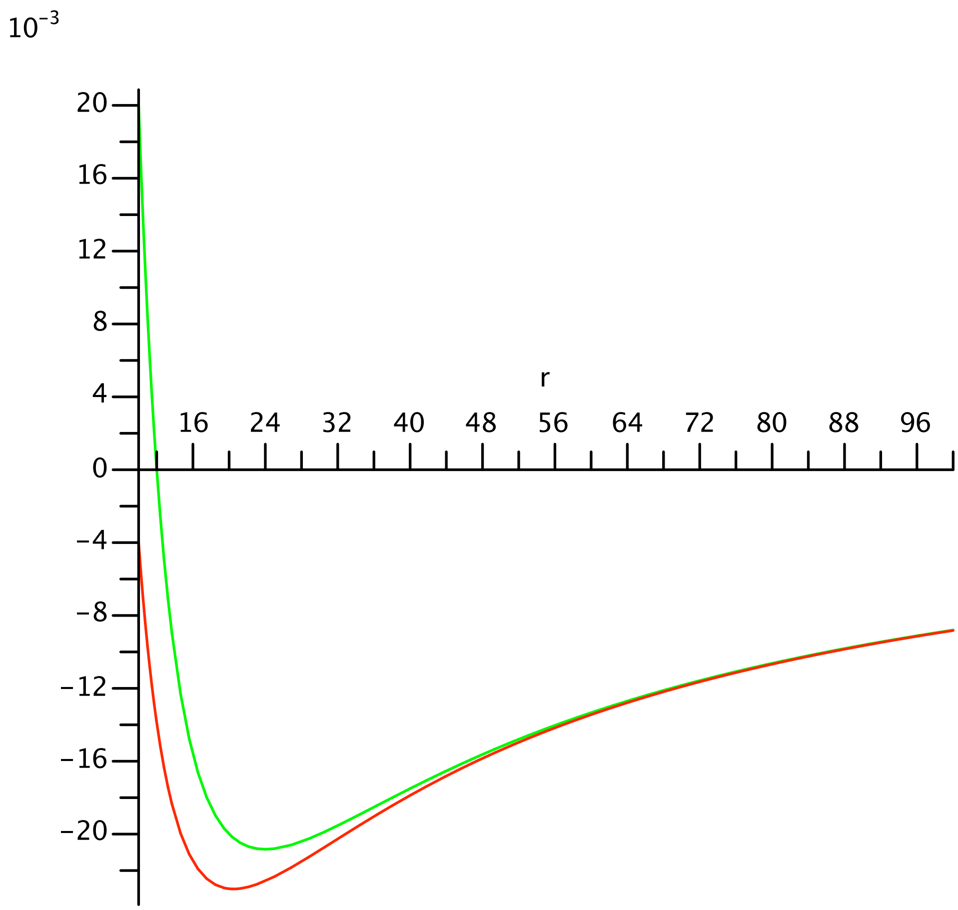
$\Rightarrow$ <u>no</u> circular orbits with $R < 3M$.

```
> Ve := -M/r + L^2/(2*r^2) - M*L^2/r^3;
  Vn := -M/r + L^2/(2*r^2);
```

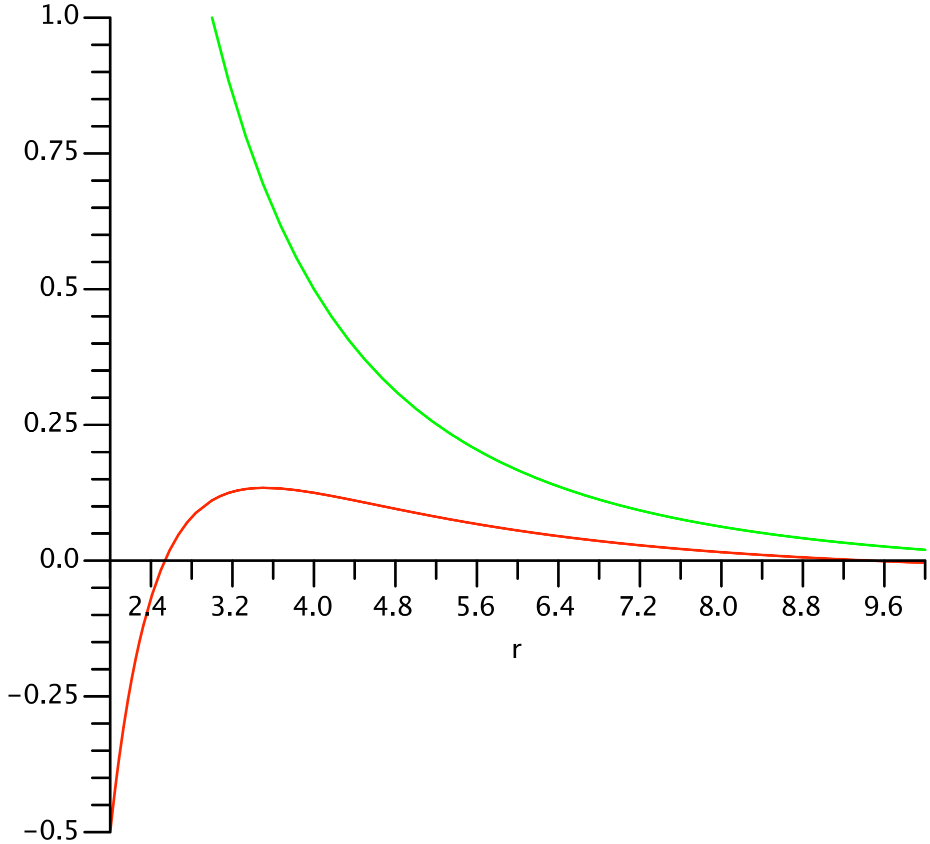
$$Ve := -\frac{M}{r} + \frac{1}{2}\,\frac{L^2}{r^2} - \frac{M L^2}{r^3}$$

$$Vn := -\frac{M}{r} + \frac{1}{2}\,\frac{L^2}{r^2} \tag{1}$$

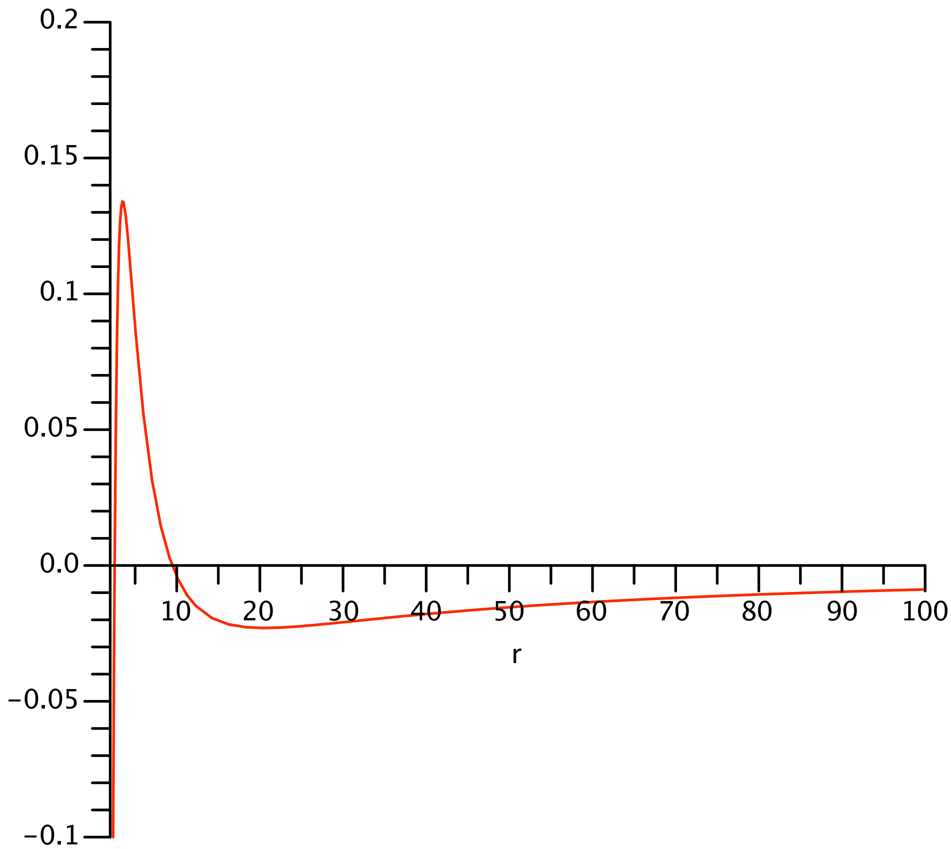
```
> plot(subs({M=1, L=sqrt(24)}, [Ve, Vn]), r=10..100);
```



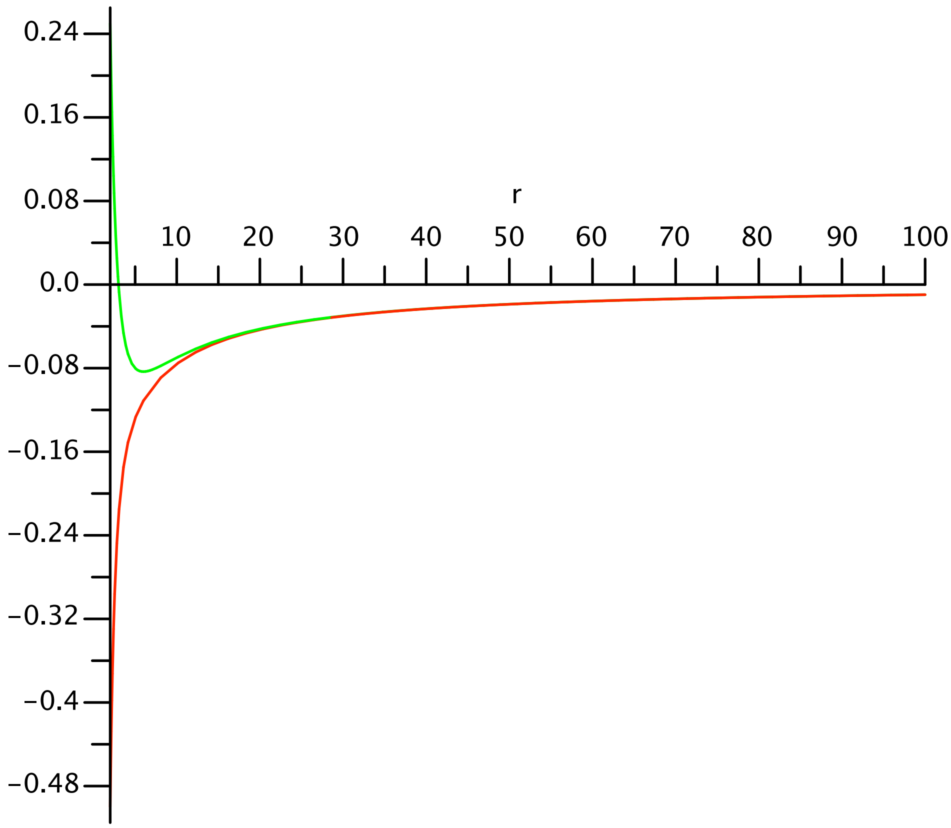

```
> plot(subs({M=1, L=sqrt(24)}, [Ve, Vn]), r=2..10, -0.5..1);
```




```
> plot(subs({M=1, L=sqrt(24)}, [Ve]), r=2..100, -0.1..0.2);
```




```
> plot(subs({M=1, L=sqrt(6)}, [Ve, Vn]), r=2..100);
```


0.24
0.16
0.08
0.0
-0.08
-0.16
-0.24
-0.32
-0.4
-0.48
r
10
20
30
40
50
60
70
80
90
100

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