Lecture 20

Black Holes

Who can't see what?

Global structure of Schwarzschild

We suppress the angular part of the metric because it is redundant. Thus, consider

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2$$

$$= \underbrace{\left(1-\frac{2M}{r}\right)}_{\Omega^2}\underbrace{\left[-dt^2 + \left(1-\frac{2M}{r}\right)^{-2}dr^2\right]}_{d\overset{\circ}{s}{}^2}$$

It is useful to introduce the Regge-Wheeler tortoise coordinate r_*

$$dr_* = \left(1-\frac{2M}{r}\right)^{-1}dr$$

$$= dr + \frac{\frac{2M}{r}dr}{1-\frac{2M}{r}}$$

$$= dr + \frac{2M\,dr}{r-2M}$$

We can integrate this to find

$$r_* = r + 2M \ln \left[\frac{r}{2M} - 1 \right]$$

We introduce the null coordinates

$$u := t - r_* \qquad -\infty < u < \infty$$

$$v := t + r_* \qquad -\infty < v < \infty$$

of the conformal (flat) metric

$$d\mathring{s}^2 = -dt^2 + dr_*^2 = -du\,dv .$$

The physical metric is

$$ds^2 = \left(1 - \frac{2M}{r}\right) \cdot -du\,dv$$

$$= -\frac{2M}{r}\left(\frac{r}{2M} - 1\right) du\,dv$$

$$= -\frac{2M}{r} e^{(r_* - r)/2M} du\,dv$$

$$= -\underbrace{\frac{2M}{r} e^{-r/2M}}_{\text{regular at } r=2M} \underbrace{e^{(v-u)/4M} du\,dv}_{\text{differentials}}$$

So, we reparameterize the null cooordinates to define

$$U := -4M e^{-u/4M} \qquad -\infty < U < 0$$

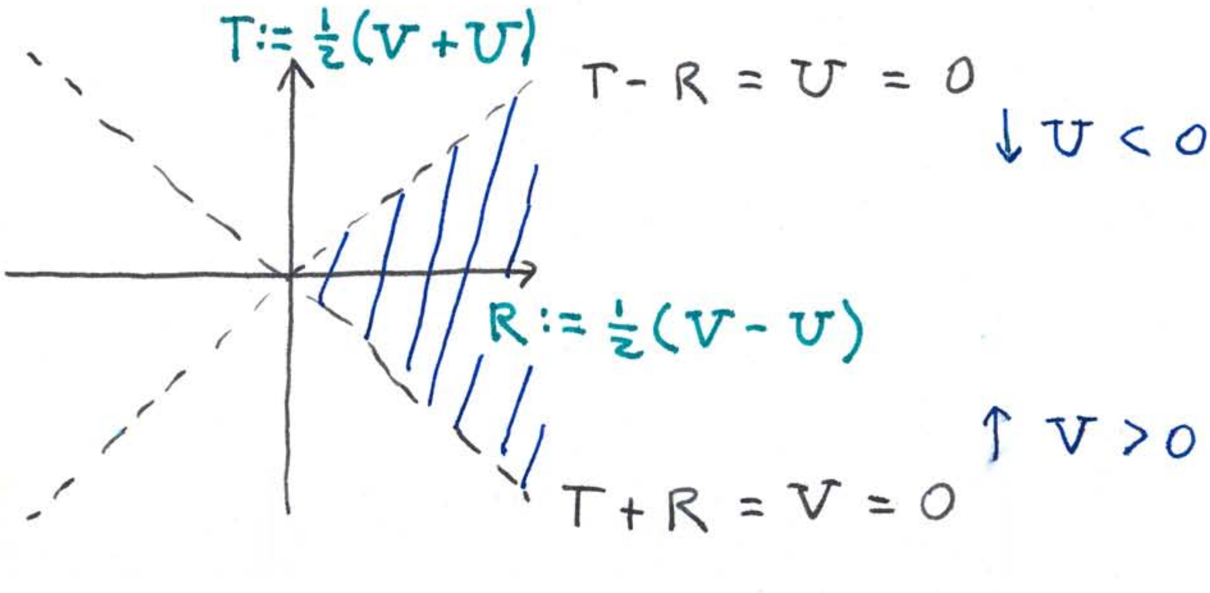
$$V := 4M e^{v/4M} \qquad 0 < V < \infty$$

In these coordinates, the physical metric becomes

$$ds^2 = \frac{2M}{r} e^{-r/2M} \cdot -dU\,dV$$

$$= \frac{2M}{r} e^{-r/2M} \left(-dT^2 + dR^2\right)$$

where we have defined



So, the exterior region $(r > 2M)$ is mapped conformally into the Rindler wedge in flat Minkowski spacetime via the diffeomorphism

$$T = \frac{1}{2}(V + U)$$

$$= \frac{1}{2}\left(4M e^{v/4M} - 4M e^{-u/4M}\right)$$

$$= 2M\left(e^{(t+r_*)/4M} - e^{-(t-r_*)/4M}\right)$$

$$= 4M \sinh\frac{t}{4M} \cdot e^{r/4M}\left[\frac{r}{2M} - 1\right]^{1/2}$$

$$R = 4M \cosh\frac{t}{4M} \cdot e^{r/4M}\left[\frac{r}{2M} - 1\right]^{1/2}$$

$$\Rightarrow \quad t = 4M \tanh^{-1}\frac{T}{R}$$

$$e^{r/2M}\left[\frac{r}{2M} - 1\right] = \frac{R^2 - T^2}{16M^2}$$

The Interior Region

The Schwarzschild metric also makes perfect sense in the interior region ($r < 2M$)

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2$$

$$= \underbrace{\left(\frac{2M}{r}-1\right)}_{\Omega^2}\underbrace{\left[-\left(1-\frac{2M}{r}\right)^{-2}dr^2 + dt^2\right]}_{d\mathring{s}^2}$$

Note that in this case, the radial coordinate r plays the role of time in the metric signature.

$r = \text{const.}$ is a space-like hypersurface.

We again define the tortoise coordinate, though now

$$r_* = r + 2M \ln\left[1 - \frac{r}{2M}\right] \qquad \begin{array}{l} 0 < r < 2M \\ 0 > r_* > -\infty \end{array}$$

We <u>assume</u> that r_* increases to the <u>future</u> and define the null coordinates

$$u := r_* - t \qquad -\infty < u < \infty$$

$$v := r_* + t \qquad -\infty < v < \infty$$

The physical metric becomes

$$ds^2 = \left(\frac{2M}{r} - 1\right) \cdot -du\,dv$$

$$= -\frac{2M}{r}\left(1 - \frac{r}{2M}\right) du\,dv$$

$$= -\frac{2M}{r} e^{(r_* - r)/2M}\, du\,dv$$

$$= -\frac{2M}{r} e^{-r/2M} e^{(v+u)/4M}\, du\,dv$$

This suggests that we define

$$U := 4M e^{u/4M} \qquad 0 < U < \infty$$

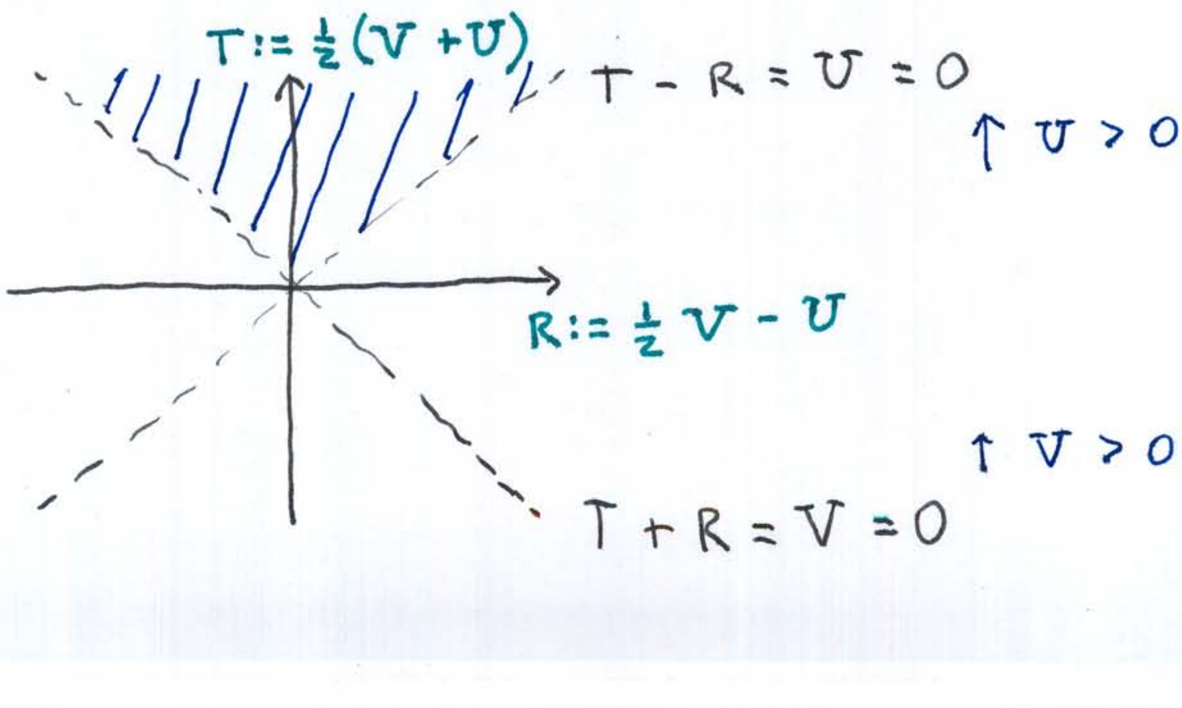
$$V := 4M e^{v/4M} \qquad 0 < V < \infty$$

The physical metric becomes

$$ds^2 = \frac{2M}{r} e^{-r/2M} \cdot -dU\,dV$$

$$= \frac{2M}{r} e^{-r/2M} \left(-dT^2 + dR^2\right)$$

This is the same metric we found before, but restricted to a different wedge:



So the interior region $(r < 2M)$ is mapped conformally into the future light cone of the origin in Minkowski spacetime via the diffeomorphism

$$T = \frac{1}{2}(V + U)$$
$$= \frac{1}{2}\left(4M e^{v/4M} + 4M e^{u/4M}\right)$$
$$= 2M\left(e^{(r_* + t)/4M} + e^{(r_* - t)/4M}\right)$$
$$= 4M \cosh\frac{t}{4M} \cdot e^{r/4M}\left[1 - \frac{r}{2M}\right]^{1/2}$$

$$R = 4M \sinh\frac{t}{4M} \cdot e^{r/4M}\left[1 - \frac{r}{2M}\right]^{1/2}$$

$$\Rightarrow \quad t = 4M \tanh^{-1}\frac{R}{T}$$

$$e^{r/2M}\left[1 - \frac{r}{2M}\right] = \frac{T^2 - R^2}{16M^2}$$

same as exterior

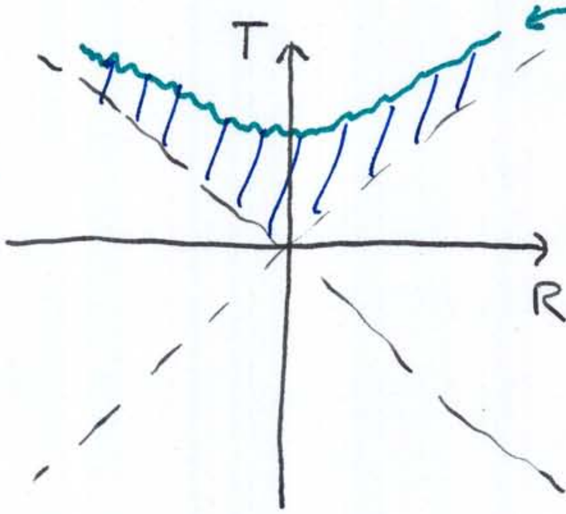
Unlike the exterior case, there is a genuine boundary to the conformal mapping in the interior

$$ds^2 = \underbrace{\frac{2M}{r} e^{-r/2M}}_{\uparrow} \left(-dT^2 + dR^2\right)$$

conformal factor is singular at $r=0$

Thus, we must have

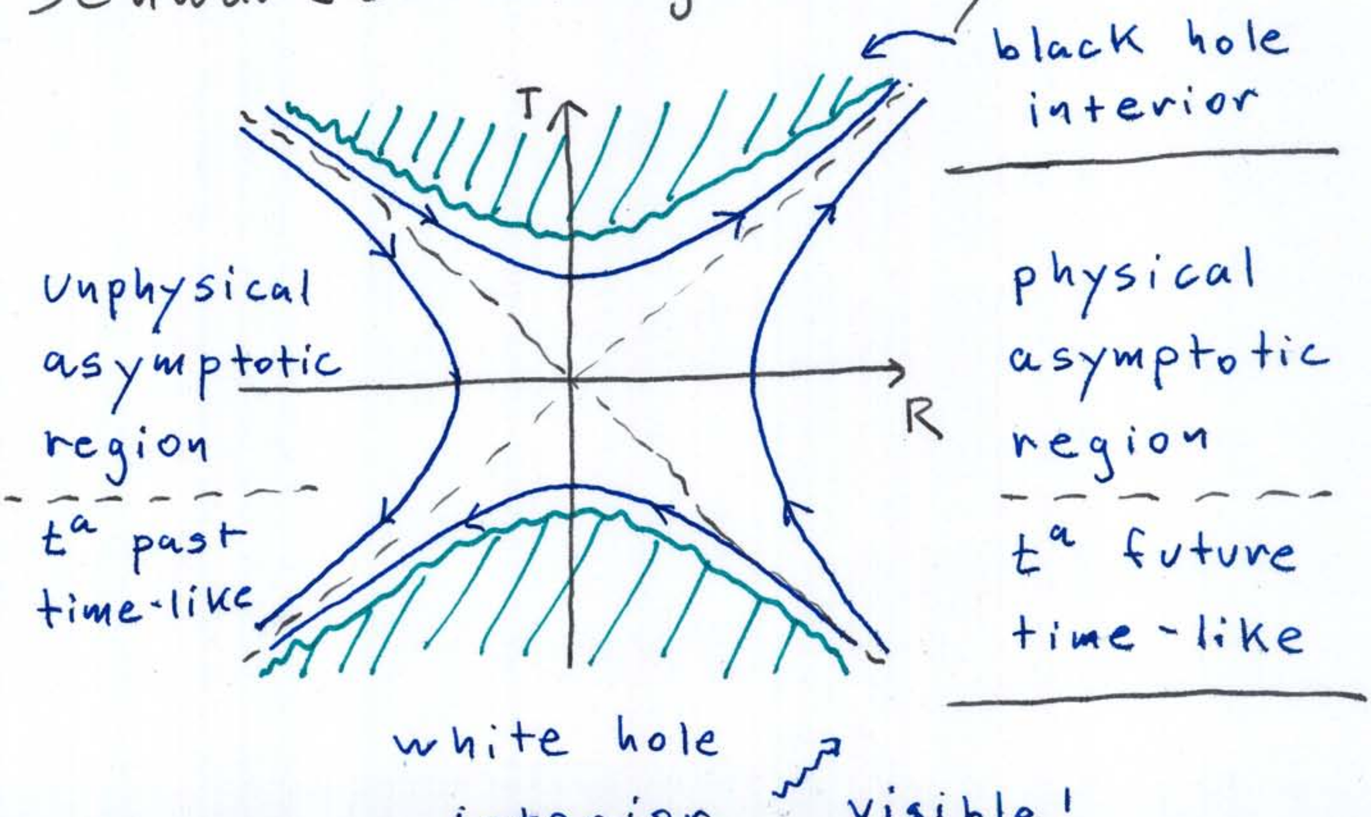
$$T^2 - R^2 = 16M^2 e^{r/2M}\left[1 - \frac{r}{2M}\right] < 16M^2$$



space-like singularity at $r=0$

The Kruskal Extension

If we repeat these analyses <u>assuming</u> that the appropriate coordinate (t, r_*) increases to the <u>past</u> in the conformal metric, we find that the other two wedges are also conformal to regions of the Schwarzschild geometry:



Eddington - Finkelstein Coordinates

Causality demands that objects starting in the physical asymptotic region must either remain there or fall into the black hole interior. The Eddington-Finkelstein coordinates (v, r) are particularly useful to study this entire physical region.

$$v = t + r_* \quad \Rightarrow \quad t = v - r_*$$

$$dt = dv - dr_* = dv - \left(1 - \frac{2M}{r}\right)^{-1} dr$$

$$\Rightarrow ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2\, dv\, dr$$

The Eddington - Finkelstein metric

$$ds^2 = -\left(1-\frac{2M}{r}\right)dv^2 + 2\,dv\,dr$$

is regular and has regular inverse $(\det g = -1)$ all the way down to the singularity.

The coordinate basis vectors are related by

$$dt = dv - \left(1-\frac{2M}{r}\right)^{-1} dr$$

$$\left(\frac{\partial}{\partial t}\right)_r = \left(\frac{\partial}{\partial v}\right)_r \leftarrow \text{Killing vector}$$

$$\left(\frac{\partial}{\partial r}\right)_t = \left(\frac{\partial}{\partial r}\right)_v + \left(\frac{\partial v}{\partial r}\right)_t \left(\frac{\partial}{\partial v}\right)_r$$

$$= \underbrace{\left(\frac{\partial}{\partial r}\right)_v}_{\text{always null}} + \underbrace{\left(1-\frac{2M}{r}\right)^{-1}}_{\text{singular at } r=2M} \left(\frac{\partial}{\partial v}\right)_r$$

What is a Black Hole?

- The static observers "at infinity" in the Schwarzschild geometry are physically preferred because they are also inertial.

- The interior of the Schwarzschild black hole is physically defined by the property that light rays cannot escape to be seen by static observers in the physical asymptotic region.

- Generally, spacetime will not be static, but an isolated system should admit inertial observers "at infinity" who see it at rest.

Definition of a Black Hole

- A spacetime is said to be <u>asymptotically flat</u> if, in a mathematically precise sense, the geometry approaches that of Minkowski spacetime as one goes "to infinity."
(boundary conditions)

- An asymptotically flat metric is said to describe a <u>black hole</u> if there is a <u>trapped region</u> from which null geodesics cannot escape "to infinity.

- The boundary of that trapped region is called the <u>event horizon</u>.

Adding Infinity

To describe asymptotically flat spacetimes, we add new boundary points to spacetime corresponding to the various ways a curve can run off to infinity. This process is called conformal compactification because infinity is not well-defined in the physical metric.

$$ds^2 = \Omega^2 d\mathring{s}^2$$

ds^2: physical metric, boundary at infinite distance.

Ω: conformal factor that diverges on the boundary

$d\mathring{s}^2$: conformal metric on a manifold with boundary

One-Point Compactification

An important example involves the Euclidean plane $\mathbb{E}^2$ and its compactification, the sphere S^2. This may be familiar from complex variables (Riemann sphere)

$$ds^2 = dr^2 + r^2 d\theta^2$$

<u>Choose</u>: $\Omega = \frac{1+r^2}{2}$

$$\overset{0}{ds}{}^2 = \Omega^{-2} ds^2$$
$$= \frac{4dr^2}{(1+r^2)^2} + \frac{4r^2 d\theta^2}{(1+r^2)^2}$$
$$= d\psi^2 + \sin^2\psi \, d\theta^2$$

where $r = \tan(\frac{1}{2}\psi)$. The inverse of this <u>conformal embedding</u> is the stereographic projection.

Minkowski Compactified

The physical Minkowski metric is

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$$

$$= -du\,dv + \tfrac{1}{4}(v-u)^2 d\Omega^2$$

$$u := t - r \qquad t = \tfrac{1}{2}(v+u)$$

$$v := t + r \qquad r = \tfrac{1}{2}(v-u)$$

We choose the conformal factor

$$\Omega^2 = \tfrac{1}{4}(1+u^2)(1+v^2)$$

This gives the conformal metric

$$d\mathring{s}^2 = \frac{-4\,du\,dv}{(1+u^2)(1+v^2)} + \frac{(v-u)^2 d\Omega^2}{(1+u^2)(1+v^2)}$$

Note that the components here are well-behaved as $u, v \to \infty$.

We now bring infinity to a finite coordinate distance by setting

$$u = \tan U \qquad du = (1+u^2)\, dU$$

$$v = \tan V \qquad dv = (1+v^2)\, dV$$

Then we find

$$d\mathring{s}^2 = -4\, dU\, dV + \frac{(\tan V - \tan U)^2}{\sec^2 U \sec^2 V}\, d\Omega^2$$

$$= -4\, dU\, dV + \sin^2(V-U)\, d\Omega^2$$

Finally, we introduce coordinates

$$\tau := U + V \qquad \psi := V - U$$

$$\rightsquigarrow\ d\mathring{s}^2 = -d\tau^2 + d\psi^2 + \sin^2\psi\, d\Omega^2$$

This metric describes the round unit 3-sphere S^3 crossed with time τ. It is a cylinder.