Lecture 22

Weak Gravitational

Waves

How does spacetime oscillate?

Review of Weak Fields

We expand the metric to first order in a suitable perturbation parameter λ:

$$g_{ab} = \eta_{ab} + \lambda \dot{g}_{ab} + \mathcal{O}(\lambda^2)$$

physical metric — Minkowski background — perturbation

The first-order field equation takes its simplest form in terms of the trace-reversed perturbation field

$$h_{ab} := \left(\delta_a^m \delta_b^n - \tfrac{1}{2}\eta_{ab}\eta^{mn}\right)\dot{g}_{mn}$$

$$\partial_c \partial^c h_{ab} - 2\,\partial_{(a}\partial^c h_{b)c} + \eta_{ab}\partial^c\partial^d h_{cd}$$

$$= -16\pi\, t_{ab} \leftarrow \text{source perturbation}$$

Like the Maxwell equations, the post-Minkowski equations are both over- and under-determined:

- The left side of the field equation <u>always</u> has zero divergence, so the source must obey $\partial^a t_{ab} = 0$.
- When a solution h_{ab} exists, it is determined only up to a gauge transformation

$$h_{ab} \to \tilde{h}_{ab} = h_{ab} + 2\partial_{(a}\phi_{b)} - \eta_{ab}\partial^c\phi_c$$

Note that the gauge transformations are exactly the infinitesimal diffeomorphisms of spacetime

$$\dot{\tilde{g}}_{ab} = \dot{g}_{ab} + 2\partial_{(a}\phi_{b)} = \dot{g}_{ab} + \mathcal{L}_\phi \eta_{ab}$$

As with the Maxwell equations, we can solve the post-Minkowski equations most easily if we fix the <u>de Donder ("Lorentz") gauge</u> $\partial^a h_{ab} = 0$. Every perturbation field can be put in this gauge:

$$\partial^a \tilde{h}_{ab} = \partial^a h_{ab} + 2\partial^a \partial_{(a} \phi_{b)} - \partial_b \partial^c \phi_c$$

$$= \partial^a h_{ab} + \partial^a \partial_a \phi_b$$

$\rightsquigarrow$ solve $\partial^a \partial_a \phi_b = -\partial^a h_{ab}$

The gauge-fixed field is not quite unique. There are residual gauge transformations

$$h_{ab} \to \tilde{h}_{ab} = h_{ab} + 2\partial_{(a}\phi_{b)} - \eta_{ab}\partial^c \phi_c$$

<u>with</u> $\partial^a \partial_a \phi_b = 0$.

Vacuum Solutions

If the first-order source t_{ab} vanishes we must solve the simultaneous equations

$$\partial^c \partial_c h_{ab} = 0 \qquad \partial^a h_{ab} = 0$$

This is most easily done in the Fourier space of the Minkowski background. We write

$$h_{ab}(x) = \int \frac{d^4k}{(2\pi)^2} e^{ik\cdot x} \hat{h}_{ab}(k)$$

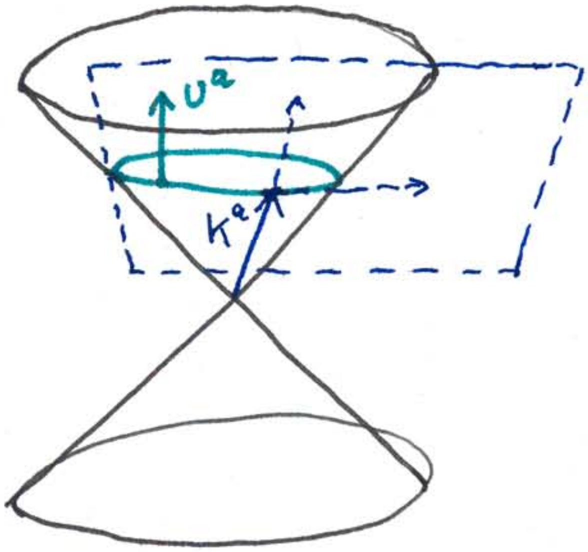
$$\hat{h}_{ab}(k) := \int \frac{d^4x}{(2\pi)^2} e^{-ik\cdot x} h_{ab}(x)$$

Note that we have implicitly used the flatness of the background to integrate a tensor.

In Fourier space, as usual, our equations become algebraic

$$-K^c K_c \hat{h}_{ab} = 0 \qquad i K^a \hat{h}_{ab} = 0$$

$\hat{h}_{ab}$ is non-zero only on the light cone $K \cdot K = 0$. | $\hat{h}_{ab}$ only has components along the cone



tangent plane at K

$$\hat{h}_{ab} = \hat{h}^{nn} K_a K_b \leftarrow 1$$
$$+ 2 \hat{h}^{ns}_{(a} K_{b)} \leftarrow 2$$
$$+ \hat{h}^{ss}_{ab} \leftarrow 3$$

If we choose any time-like U^a, the decomposition of $\hat{h}_{ab}$ into <u>null</u> and <u>spatial</u> pieces is <u>unique</u>.

$$\hat{h}_{ab} U^b = \left(\hat{h}^{nn} K_a + \hat{h}^{ns}_a\right) K_b U^b$$

Transverse - Traceless Gauge

A residual gauge transformation acts in Fourier space by

$$\hat{h}_{ab} \to \hat{\tilde{h}}_{ab} = \hat{h}_{ab} + 2 i K_{(a} \hat{\phi}_{b)} - \eta_{ab} i K^c \hat{\phi}_c$$

with $-K^a K_a \hat{\phi}_b = 0$.

The four components of $\hat{\phi}_b$ can modify the three null components of $\hat{h}_{ab}$ and the one trace component of the spatial part.

$\rightsquigarrow$ The traceless part of $\hat{h}^{ss}_{ab}$ is gauge-invariant!

Its two components correspond to the two polarization states of gravitational radiation.

Given an inertial frame U^a, we can exhaust the residual gauge freedom by eliminating all but the gauge-invariant components. This defines the <u>transverse-traceless gauge</u>.

$$4 \to K^a \hat{h}^{TT}_{ab} = 0 \qquad \partial^a h^{TT}_{ab} = 0$$

$$3 \to U^a \hat{h}^{TT}_{ab} = 0 \qquad U^a h^{TT}_{ab} = 0$$

$$1 \to \eta^{ab} \hat{h}^{TT}_{ab} = 0 \qquad \eta^{ab} h^{TT}_{ab} = 0$$

↑ Fourier space ↑ spacetime.

For a plane wave propagating in the +z-direction, we can break $\hat{h}^{TT}_{ab}$ into polarizations

$$\hat{h}^{TT}_{ab} = \hat{h}^{+} (e^x_a e^x_b - e^y_a e^y_b) \leftarrow \text{"plus"}$$

$$+ \hat{h}^{\times}\, 2 e^x_{(a} e^y_{b)} \leftarrow \text{"cross"}$$

Motion of Test Particles

Putting a plane wave in transverse-traceless gauge introduces a preferred frame u^a on spacetime. How does a test particle at rest in this frame move under the influence of the wave?

$$u^a \nabla_a u^c = -u^a u^b \dot{\Gamma}_{ab}{}^c$$

$$= -\tfrac{1}{2} g^{cm} u^a u^b (2\partial_{(a} \dot{g}_{b)m} - \partial_m \dot{g}_{ab})$$

$$= -\tfrac{1}{2} \eta^{cm} u^a u^b (2\partial_{(a} h^{TT}_{b)m} - \partial_m h^{TT}_{ab})$$

$$= -\tfrac{1}{2} \eta^{cm} (2\partial_0 h^{TT}_{0m} - \partial_m h^{TT}_{00}) = 0$$

Thus, the perturbation does not affect the motion of such test particles <u>relative to the coordinates</u>!
(This is a result of TT gauge!)

A more physical question concerns the proper distance between such particles.

Consider a spherical cloud of test particles with radius r and at rest in the Minkowski background of a TT-wave. We have seen that the coordinate expressions of the geodesics are unmodified:

$$\gamma^a(\tau) = \tau u^a + r^a$$

But the proper distance to a given test particle oscillates:

$$\|r\|^2 = r^a r^b (\eta_{ab} + \dot{g}_{ab}) = r^2 + r^a r^b h^{TT}_{ab}$$

$$= r^2 + \left[h^+ (x^2 - y^2) + h^\times (2xy)\right] e^{i\omega(t-z)}$$

$$= r^2 \left[1 + \sin^2\theta \left(h^+ \cos 2\phi + h^\times \sin 2\phi\right)\right] \cdot e^{i\omega(t-z)}$$

An even more physical calculation uses geodesic deviation to measure the relative acceleration of nearby geodesics in the perturbed metric:

$$\nabla_u \nabla_u \xi^d = \xi^a u^b u^c R_{abc}{}^d$$

relative acceleration

$$= \xi^a u^b u^c \dot{R}_{abc}{}^d$$

relative displacement

background is flat

Note that the first-order Riemann curvature is gauge-invariant because the background Minkowski geometry is flat.

$$\dot{\tilde{R}}_{abc}{}^d = \dot{R}_{abc}{}^d + \mathcal{L}_\phi R_{abc}{}^d \leftarrow 0$$

Thus, we can use <u>any</u> gauge to evaluate the deviation, including the TT gauge adapted to u^a!

Working in this gauge,

$$\dot{R}_{abcd} = 2\,\partial_{[a}\dot{\nabla}_{b]cd}$$
$$= -\partial_{[a}\left(\partial_{b]}\dot{g}_{cd} + 2\,\partial_{[c}\dot{g}_{d]b]}\right)$$
$$= -2\,\partial_{[a}\partial_{[c}h^{TT}_{d]b]}$$

Since <u>any</u> contraction of h^{TT}_{ab} with v^a vanishes, many terms vanish in the geodesic deviation:

$$a_d = \xi^a v^b v^c \cdot -2\,\partial_{[a}\partial_{[c}h^{TT}_{d]b]}$$
$$= \tfrac{1}{2}\,\xi^a\,\partial_0\partial_0\,h^{TT}_{da}$$

This describes, in a completely coordinate- and gauge-invariant way, the effect of a passing linearized gravitational wave.

Non-Vacuum Solutions

In the presence of a first-order source t_{ab} satisfying the integrability condition $\partial^a t_{ab} = 0$, we must simultaneously solve

$$\partial^c \partial_c h_{ab} = -16\pi\, t_{ab} \qquad \partial^a h_{ab} = 0$$

The first is the inhomogeneous flat-space wave equation. We solve it using the retarded Green function $G_y^{ret}(x)$ since our boundary conditions will demand no incoming radiation:

$$h_{ab}(x) = -16\pi \int G_y^{ret}(x)\, t_{ab}(y)\, d^4y$$

$$G_y^{ret}(x) = \frac{-\delta(t_x - t_y - |\vec{x}-\vec{y}|)}{4\pi\, |\vec{x}-\vec{y}|}$$

(support on the future light cone of y.)

We still must check that the gauge condition $\partial^a h_{ab} = 0$ holds for this retarded solution. This can be checked easily in Fourier space:

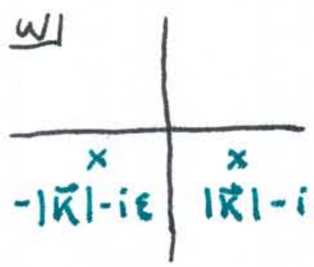
$$-K^2 \hat{h}_{ab} = -16\pi \hat{t}_{ab}$$

$$\Rightarrow \hat{h}_{ab} = \frac{-16\pi \hat{t}_{ab}}{\omega^2 - |\vec{K}|^2} + \hat{c}_{ab}$$

poles on contour, must be regularized

support on cone, homogeneous sol'n.

$$h_{ab}(x) = \int \frac{d\omega\, d^3K}{(2\pi)^2} \hat{h}_{ab}(K)\, e^{-i\omega t + i\vec{K}\cdot\vec{x}}$$



$t > 0 \Rightarrow$ close contour in <u>lower</u> half-plane.

$$\Rightarrow \hat{h}^{ret}_{ab}(K) = \frac{-16\pi \hat{t}_{ab}(K)}{(\omega + i\varepsilon)^2 - |\vec{K}|^2}$$

$$\Rightarrow K^a \hat{h}^{ret}_{ab}(K) = 0$$