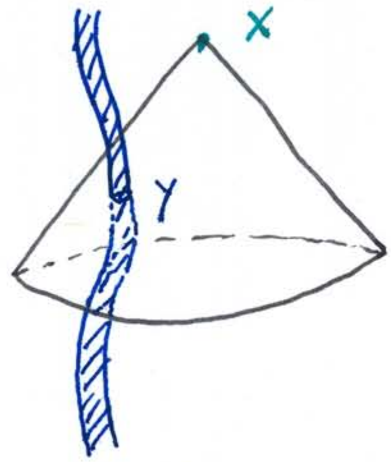
Compact Slow-Motion Sources



The retarded field at the point x is given by an integral over its past light cone.

We make two approximations:

Compact Source: $|\vec{x}| \gg R_{source}$

We are in the far-field region (wave zone, etc.) where the source can be treated as a point.

Slow Motion: $v_{source} \ll 1$

The source doesn't change much as the light cone sweeps across it.

$\rightsquigarrow$ Retardation effects <u>within</u> the source are negligible.

The general expression for the field is

$$h_{ab}(x) = 4\int \frac{\delta(t_x - t_y - |\vec{x}-\vec{y}|)}{|\vec{x}-\vec{y}|}\, t_{ab}(y)\, d^4y$$

In the limit of slow, compact sources, this becomes

$$h_{ab}(t,\vec{x}) \cong \frac{4}{|\vec{x}|}\int t_{ab}(t-|\vec{x}|, \vec{y})\, d^3y$$

t_{ret} at the origin

The temporal components in the center-of-mass frame are easy to calculate:

$$h_{00} = \frac{4}{r}\int t_{00}(t-r,\vec{y})\, d^3y = \frac{4M}{r}$$

$$h_{0i} = \frac{4}{r}\int t_{0i}(t-r,\vec{y})\, d^3y = \frac{4}{r} P_i = 0$$

The spatial components may be computed using a trick:

$$\int \bar{t}^{ab}\, d^3y = \int \bar{\partial}_c \bar{y}^a \cdot \bar{t}^{cb}\, d^3y$$

$$= -\int \bar{y}^a \bar{\partial}_c \bar{t}^{cb}\, d^3y$$

$$= \int \bar{y}^a \partial_0 \bar{t}^{0b}\, d^3y$$

$$= \frac{d}{dt}\int \bar{y}^{(a} \bar{t}^{b)0}\, d^3y$$

$$\int \bar{y}^{(a} \bar{t}^{b)0}\, d^3y = \int \bar{y}^{(a} \bar{\partial}_c \bar{y}^{b)} \cdot \bar{t}^{c0}\, d^3y$$

$$= -\int \bar{y}^{(b} \bar{\partial}_c (\bar{y}^{a)} \bar{t}^{c0})\, d^3y$$

$$= -\int \left[\bar{y}^{(b} \bar{\partial}_c \bar{y}^{a)} \cdot \bar{t}^{c0} + \bar{y}^{(b}\bar{y}^{a)} \bar{\partial}_c \bar{t}^{c0}\right] d^3y$$

$$= \int \bar{y}^a \bar{y}^b \partial_0 t^{00}\, d^3y - \int \bar{y}^{(b} \bar{t}^{a)0}\, d^3y$$

$$\Rightarrow \int \bar{t}^{ab}\, d^3y = \frac{1}{2}\frac{d^2}{dt^2}\int t^{00}\, \bar{y}^a \bar{y}^b\, d^3y$$

$$\bar{\partial}_c \, \bar{t}^{cb} + \partial_0 \, \bar{t}^{0b} = 0$$

The last integral is the tensor <u>quadrupole moment</u> of the source energy density $\rho := t^{00}$.

$$\bar{h}_{ab} = \frac{2}{r}\left[\frac{d^2}{dt^2}\bar{I}_{ab}(t)\right]_{ret}$$

$$\bar{I}_{ab} := \int T_{00}(\vec{y})\,\bar{y}_a\,\bar{y}_b\,d^3y$$

quadrupole formula

We can restore units to this as follows:

$$T^{00} \sim \frac{E}{L^3} \Rightarrow I \sim T^{00}L^5 \sim EL^2$$

$$\Rightarrow \ddot{I} \sim \frac{EL^2}{T^2} \Rightarrow \frac{\ddot{I}}{r} \sim \frac{EL}{T^2}$$

$$\Rightarrow \frac{G}{c^4}\frac{\ddot{I}}{r} \sim \frac{GML}{L^2} \sim \frac{E}{M} \sim c^2$$

$$\Rightarrow \bar{h}_{ab} = \frac{2G}{c^6 r}\left[\frac{d^2}{dt^2}\bar{I}_{ab}(t)\right]_{ret}$$

small!

Gaussian Integrals

1) $\dot{G}^{ac} = \frac{1}{2} \partial_b \partial_d H^{abcd}$

$$H^{abcd} := -4\, h^{[a[c} \eta^{d]b]}$$

$$= H^{cdab}$$

$$H^{[abc]d} = 0$$

$$\partial_a \dot{G}^{ac} = \frac{1}{2} \partial_a \partial_b \partial_d H^{[ab]cd} = 0$$

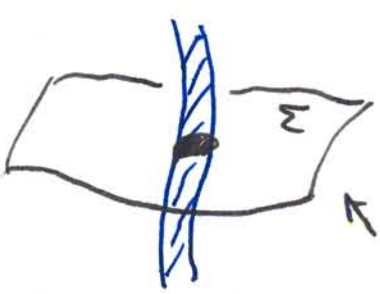
2) Four - Momentum

$$P^a = -\int_\Sigma \dot{T}^{ac}\, \hat{n}_c\, d\Sigma$$

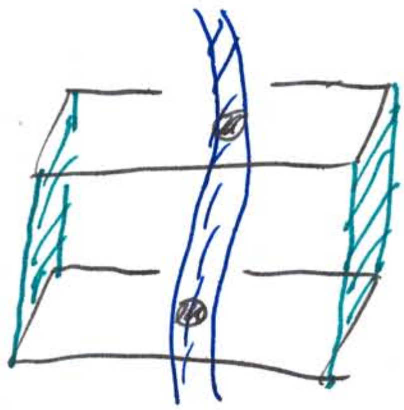
volume element

spatial slice

future-directed unit normal



$\dot{T}^{ac} = 0$ on boundary.



$$P^a = -\frac{1}{16\pi}\int_\Sigma \partial_b \partial_d H^{abcd} \cdot \hat{n}_c \, d\Sigma$$

$$= -\frac{1}{16\pi}\int_\Sigma \bar{\partial}_d \left(\hat{n}_c \partial_b H^{abcd}\right) d\Sigma$$

$$= -\frac{1}{16\pi}\oint_S \hat{n}_c \partial_b H^{abcd} \, \hat{r}_d \, dS$$

unit outward-directed normal to $S = \partial\Sigma$

$$\hat{n}_i \hat{r}_j \partial_b H^{abij} = \hat{n}_{[i} \hat{r}_{j]} \cdot -4 \partial_b h^{i[a} \eta^{b]j}$$

$$= -2 \hat{n}_{[i} \hat{r}_{j]} \partial_b \left(2\dot{g}^{i[a} \eta^{b]j} - \dot{g} \eta^{i[a} \eta^{b]j} \right)$$

$$= -2 \hat{n}_{[i} \hat{r}_{j]} \left(2\eta^{ic} \eta^{d[a} \eta^{b]j} - \eta^{cd} \eta^{ia} \eta^{bj} \right)$$

$\partial_b \dot{g}_{cd}$

$$\left(-\eta^{jc} \eta^{da} \eta^{bi} - \eta^{ic} \eta^{db} \eta^{aj} - \eta^{cd} \eta^{ia} \eta^{bj} \right)$$

$$= -\left(\eta^{ib} \eta^{jc} \eta^{ad} + \eta^{ic} \eta^{ja} \eta^{bd} + \eta^{ia} \eta^{jb} \eta^{cd} \right)$$

$$\hat{n}_i \hat{r}_j \partial_b H^{abij} = -6 \hat{r}^{[a} \hat{n}^{b} \eta^{c]d} \partial_b \dot{g}_{cd}$$

$$\eta^{cd} = -\hat{n}^c \hat{n}^d + q^{ab}$$

2-sphere metric

$$= -\hat{n}^c \hat{n}^d + \hat{r}^a \hat{r}^b + \sigma^{ab}$$

$$\hat{n}_i \hat{r}_j \partial_b H^{abij} = -6 \hat{r}^{[a} \hat{n}^{b} \sigma^{c]d} \partial_b \dot{g}_{cd}$$

$$P^a = \frac{3}{8\pi} \oint_S \hat{r}^{[a} \hat{n}^b \epsilon^{c]d} \partial_b \dot{g}_{cd}$$

$$\uparrow$$

$$E = -\hat{n}_a P^a$$

$$= -\frac{1}{8\pi} \oint_S \hat{r}^{[b} \epsilon^{c]d} \partial_b \dot{g}_{cd}$$

$$= \dot{-} \frac{1}{8\pi} \oint_S \hat{r}^{[b} \epsilon^{c]d} \bar{\partial}_b \dot{q}_{cd}$$

$$\uparrow$$

$$V\cancel{A} - 2\Phi\, \partial_c r \cdot \partial_d r$$

$$= \frac{1}{4\pi} \oint_S \hat{r}^{[b} \epsilon^{c]d} \bar{\partial}_b \left(\Phi\, \partial_c r \cdot \partial_d r \right)$$