Lecture 24

Energy Loss to

Gravitational Radiation.

Gaussian Integrals

Last Time: The total energy-momentum p^a of the source t^{ab} of the post-Minkowskian field $\dot{g}_{ab}$ is given by

$$p^a = \frac{3}{8\pi} \oint_S \hat{n}^{[a} \hat{n}^{b} \sigma^{c]d} \partial_b \dot{g}_{cd} \, dS$$

Here, S is the spherical outer boundary of a spatial slice Σ through spacetime, $\hat{n}^a$ is the time-like normal to Σ, $\hat{r}^a$ is the space-like normal to S within Σ, and σ^{ab} is the two-dimensional intrinsic metric on S.

Newtonian Limit

The energy E of the source is given by the component

$$E := -\hat{n}_a P^a = -\frac{1}{8\pi}\oint_S \hat{r}^b \sigma^{cd}\, \bar{\partial}_{[b}\, \dot{q}_{c]d}\, dS$$

Here, $\bar{\partial}_a$ is the intrinsic flat derivative operator on the spatial slice Σ, which we have now assumed to be inertial, and $\dot{q}_{ab}$ is the perturbation in the spatial metric.

When $\dot{q}_{ab} = -2\Phi q_{ab}$ in the Newtonian limit, we find

$$E = \frac{1}{4\pi}\oint_S \hat{r}^{[b}\sigma^{c]d}\, q_{cd}\, \bar{\partial}_b \Phi\, dS$$

$$= \frac{1}{4\pi}\oint_S \hat{r}^b\, \bar{\partial}_b \Phi\, dS = M \leftarrow \text{Newtonian result.}$$

Gauge Invariance

To make sense physically, the energy E must not vary with the gauge of the perturbation field $\dot{g}_{ab}$. Since the integral is linear in the perturbation, we can check this by setting $\dot{g}_{ab} = 2\partial_{(a}\phi_{b)}$

$$\Delta E = \frac{-1}{8\pi}\oint_S \hat{r}^b \sigma^{cd}\, \bar{\partial}_{[b}\left(\bar{\partial}_{c]}\bar{\phi}_d + \bar{\partial}_{|d|}\bar{\phi}_{c]}\right) dS$$

(spatial projection is constant)

$$= \frac{-1}{8\pi}\oint_S \hat{r}^b \sigma^{cd}\, \bar{\partial}_d\, \bar{\partial}_{[b}\bar{\phi}_{c]}\, dS$$

$$= \frac{-1}{8\pi}\oint_S \sigma^{cd}\, \bar{\partial}_d\left(\hat{r}^b \bar{\partial}_{[b}\bar{\phi}_{c]}\right) + K^{bc}\bar{\partial}_{[b}\bar{\phi}_{c]}$$

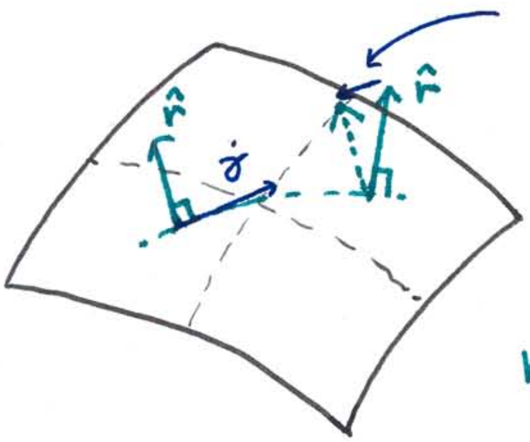
extrinsic curvature of S, $= K^{(bc)}$

$$= \frac{-1}{8\pi}\oint_S D^c\left(\hat{r}^b \bar{\partial}_{[b}\bar{\phi}_{c]}\right) dS = 0$$

intrinsic divergence on S

Extrinsic Curvature

Let S be a two-dimensional submanifold of a three-dimensional Riemannian manifold (Σ, q_{ab}).



extrinsic curvature

$$K^{ab} := -\sigma^{am}\,\bar{\partial}_m \hat{r}^b$$

- K^{ab} is symmetric:

$$K^{[ab]} = -\left(q^{m[a} - \hat{r}^m \hat{r}^{[a}\right)\bar{\partial}_m \hat{r}^{b]}$$

$$= -\bar{\partial}^{[a}\hat{r}^{b]} + \hat{r}_m\Big(3\hat{r}^{[a}\bar{\partial}^m \hat{r}^{b]} \quad \leftarrow \text{H.S.O.}$$

normalization $\rightarrow$ $\quad - \hat{r}^{[b}\bar{\partial}^{a]}\hat{r}^m$

cancels $\longrightarrow$ $\quad - \hat{r}^m\bar{\partial}^{[b}\hat{r}^{a]}\Big) = 0$

- K^{ab} is tangent:

$$\hat{r}_b K^{ab} = -\tfrac{1}{2}\sigma^{am}\bar{\partial}_m\left(\hat{r}_b \hat{r}^b\right) = 0$$

Gauge Invariance Revisited

A direct argument to prove that the *entire* four-momentum integral

$$p^a = \frac{3}{8\pi} \oint_S \hat{r}^{[a} \hat{n}^b \rho^{c]d} \partial_b \dot{g}_{cd} \, dS$$

is gauge-invariant is prohibitive.

However, we can write

$$p^a = -\frac{1}{8\pi} \int_\Sigma \dot{G}^{ac} \hat{n}_c \, d\Sigma$$

$$\Rightarrow \Delta p^a = \frac{-1}{8\pi} \int_\Sigma \mathcal{L}_\phi \overset{\circ}{G}{}^{ac} \hat{n}_c \, d\Sigma = 0$$

Note that both arguments are essentially *non-local*. The flux density at a point is *not* gauge-invariant (unlike $\vec{E}$ in E+M), but the net flux through a *closed* surface *is*.

Gravitational Field Energy

The previous results only let us calculate the <u>total</u> mass in spacetime. To calculate the loss of energy from a gravitating system to gravitational radiation, we must be more clever.

Define the <u>metric density</u>

$$\mathfrak{g}^{ab} := \sqrt{-\det g}\; g^{ab}$$

Note that $\det \mathfrak{g} = \sqrt{-\det g}^{\,4} \det g^{-1}$

$$= \det g =: -\mathfrak{g}^2$$

scalar volume density

Also note that

$$\dot{\mathfrak{g}} = \tfrac{1}{2}(-\det g)^{-1/2} \cdot \frac{d}{d\lambda}(-\det g)$$

$$= \tfrac{1}{2}(-\det g)^{1/2} \operatorname{tr} \dot{g} = \tfrac{1}{2} \mathfrak{g}^{mn} \dot{g}_{mn} = \tfrac{1}{2} g_{mn} \dot{\mathfrak{g}}^{mn}$$

Finally, note that

$$\dot{\mathfrak{g}}^{ab} = \tfrac{1}{2}\,\mathfrak{g}^{mn}\,\dot{g}_{mn}\,g^{ab} - \mathfrak{g}\,\dot{g}^{ab}$$

$$= -\mathfrak{g}\left[\delta^a_m\,\delta^b_n - \tfrac{1}{2}\,g^{ab}\,g_{mn}\right]\dot{g}^{mn}$$

$$= -\mathfrak{g}\,h^{ab} \leftarrow \text{trace-reversed.}$$

⌈This contravariant metric density is very useful in higher-order gravitational perturbation theory.⌋

Now introduce the <u>source density</u>

$$\tau^a{}_b := \mathfrak{g}\,T^a{}_b \leftarrow \text{"natural form"}$$

$$\rightsquigarrow \nabla_a \tau^a{}_b = \partial_a \tau^a{}_b + \Gamma_{ab}{}^c\,\tau^a{}_c$$

$$\Gamma_{ab}{}^c = -\tfrac{1}{2}\,g^{cd}\left(2\,\partial_{[a}\,g_{d]b} + \partial_b\,g_{ad}\right)$$

$$\nabla_a \tau^a{}_b = \partial_a \tau^a{}_b - \tfrac{1}{2}\,\partial_b\,g_{ac}\cdot\mathfrak{g}\,T^{ac}$$

Manipulating this a bit gives

$$\nabla_a \tau^a{}_b = \partial_a \tau^a{}_b + \tfrac{1}{2} T_{ac}\, g\, \partial_b g^{ac}$$

$$= \partial_a \tau^a{}_b + \tfrac{1}{2} T_{ac} \left(\partial_b g^{ac} - g^{ac} \cdot \tfrac{1}{2} g_{mn} \partial_b g^{mn} \right)$$

$$= \partial_a \tau^a{}_b + \tfrac{1}{2} T_{ac} \left(\delta^a_m \delta^c_n - \tfrac{1}{2} g^{ac} g_{mn} \right) \partial_b g^{mn}$$

Now suppose that $g_{ab} = g_{ab}(\lambda)$ solves the Einstein equation with source $T_{ab}(\lambda)$, and that $g_{ab}(0) = \eta_{ab}$ is Minkowski spacetime. Then let the coordinate connection ∂_a above be the flat Minkowski connection:

$$0 = \partial_a \tau^a{}_b + \frac{1}{16\pi} G_{ac} \left[\delta^a_m \delta^c_n - \tfrac{1}{2} g^{ac} g_{mn} \right] \partial_b g^{mn}$$

Bianchi identity

energy loss <u>in background</u>

"gravitational field energy"

The effective gravitational field energy is clearly second-order:

$$\partial_a^* \mathcal{L}^a{}_b = \frac{1}{16\pi} G_{ac}\left[\delta^a_m \delta^c_n - \tfrac{1}{2} g^{ac} g_{mn}\right] \partial_b g^{mn}$$

$$\overset{\circ}{G}_{ac} = 0 \qquad\qquad \partial_b \overset{\circ}{g}{}^{mn} = 0$$

$$\approx \frac{1}{16\pi} \dot{G}_{ac}\left[\delta^a_m \delta^c_n - \tfrac{1}{2} g^{ac} g_{mn}\right] \partial_b\left(-\overset{\circ}{g}\, h^{mn}\right)$$

We evaluate this in de Donder gauge:

$$\dot{G}_{ij}\left[\delta^i_m \delta^j_n - \tfrac{1}{2} g^{ij} g_{mn}\right] \partial_b h^{mn}$$

$$= -\tfrac{1}{2} \partial_c \partial^c h_{ij} \cdot \left[\delta^i_m \delta^j_n - \tfrac{1}{2} g^{ij} g_{mn}\right] \partial_b h^{mn}$$

$$= -\tfrac{1}{2}\left[\delta^i_m \delta^j_n - \tfrac{1}{2} g^{ij} g_{mn}\right]\left[\partial^c\left(\partial_c h_{ij}\, \partial_b h^{mn}\right) - \partial_c h_{ij} \cdot \partial_b \partial^c h^{mn}\right]$$

$$= \tfrac{1}{2} \partial_b \left(\partial_c h_{ij} \cdot \partial^c h^{mn}\right)$$

$$\mathcal{L}^a{}_b = \frac{\overset{\circ}{g}}{32\pi}\left[\delta^i_m \delta^j_n - \tfrac{1}{2} g^{ij} g_{mn}\right]\left[\partial^a h_{ij} \cdot \partial_b h^{mn} - \tfrac{1}{2}\delta^a_b\, \partial_c h_{ij} \cdot \partial^c h^{mn}\right]$$

Comments on Gauge Invariance

This effective gravitational stress-energy tensor is certainly <u>not</u> gauge-invariant. We have

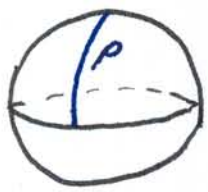
$$\partial_a \mathcal{L}^a{}_b \doteq \frac{-\mathring{g}}{16\pi} \dot{G}_{ac}\, \partial_b\, g^{ac}$$

$$\Rightarrow \partial_a (\tilde{\mathcal{L}}^a{}_b - \mathcal{L}^a{}_b) = -\frac{\mathring{g}}{8\pi} \dot{G}_{ac}\, \partial_b\, \partial^a \phi^c$$

$$= -\frac{\mathring{g}}{8\pi}\, \partial_a (\dot{G}^a{}_c\, \partial_b \phi^c)$$

Thus, at least in regions where the first-order source is non-zero, there appears to be a difficulty. Moreover, whatever the "correct" result is, it is likely quadratic in first derivatives of the field. These can be made to vanish at any one point in geodesic (Riemann normal) coordinates.

Energy and Equivalence



On a round sphere, the circumference of the equator is $2\pi r = 4\rho$.

The situation is even worse on a lumpy sphere (e.g., potato).

Why? Obviously, these surfaces are curved. But this curvature disappears at small scales, so one <u>cannot</u> explain the violation of the Euclidean relation $C = 2\pi\rho$ based solely on local analysis.

The violation emerges only <u>globally</u>, on scales comparable to the radius of curvature.

Similarly, in general relativity, it is certainly <u>true</u> that the gravitational field has energy. The weight of the solar system viewed from infinity will be less than the sum of its parts. (negative gravitational binding energy)

But the energy of the gravitational field is <u>not</u> <u>localizable</u>. It emerges only globally, on sufficiently large scales.

For gravitational waves, one must <u>average</u> over a region of several wavelengths' size to get a physical, gauge-invariant result.