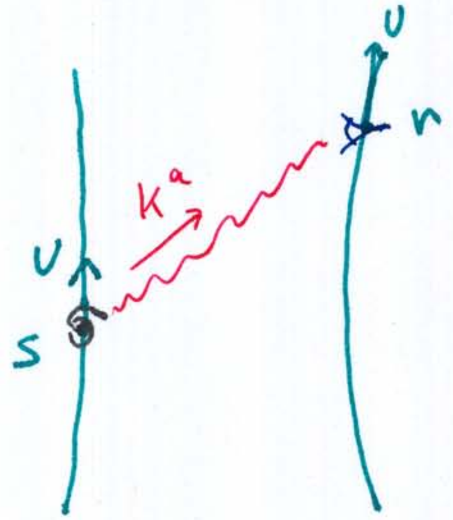
Cosmological Redshift



$$z = \frac{\lambda_r - \lambda_s}{\lambda_s}$$

$$= \frac{\omega_s}{\omega_r} - 1$$

$$= \frac{K^a U_a(\tau_s)}{K^a U_a(\tau_r)} - 1$$

Use a trick:

$$ds^2 = -d\tau^2 + a^2(\tau)\, \overset{1}{q}$$ "unit" spatial metric.

$\rightsquigarrow t$ with $dt = \frac{d\tau}{a(\tau)}$

$$ds^2 = -a^2\, dt^2 + a^2\, \overset{1}{q}$$

$$= a^2 \left(-dt^2 + \overset{1}{q} \right)$$

$\frac{\partial}{\partial t}$ = Killing.

conformal metric is cylindrical

Conformal Transformations

$$g_{ab} = \Omega^2 \mathring{g}_{ab}$$

physical

conformal (unphysical)

$$\nabla_a - \mathring{\nabla}_a \rightsquigarrow C_{ab}{}^{c}$$

$$
\begin{aligned}
0 &= \nabla_a g_{bc} \\
&= \mathring{\nabla}_a g_{bc} + C_{ab}{}^{m} g_{mc} + C_{ac}{}^{m} g_{mb}
\end{aligned}
$$

$$\Rightarrow 0 = \mathring{\nabla}_a (\Omega^2 \mathring{g}_{bc}) + 2\Omega^2 C_{a(bc)}$$

lowered with $\mathring{g}_{ab}$

$$C_{ab}{}^{c} = \mathring{g}_{ab} \mathring{\nabla} \ln \Omega - 2 \delta^{c}_{(a} \mathring{\nabla}_{b)} \ln \Omega$$

$$\mathring{K}^a \mathring{\nabla}_a \mathring{K}^c = 0$$

$$\mathring{K}^a \nabla_a \mathring{K}^c = \mathring{K}^a \mathring{\nabla}_a \mathring{K}^c - \mathring{K}^a C_{ab}{}^c \mathring{K}^b$$

$$= 2 \mathring{K}^c \mathring{K}^b \mathring{\nabla}_b \ln \Omega$$

$$\mathring{K}^a \nabla_a \mathring{K}^c + \mathring{K}^a \mathring{\nabla}_a \ln \Omega^{-2} \cdot \mathring{K}^c = 0$$

$$= \mathring{K}^a \nabla_a \mathring{K}^c + \mathring{K}^a \frac{\nabla_a \Omega^{-2}}{\Omega^{-2}} \mathring{K}^c = 0$$

$$= \frac{\mathring{K}^a}{\Omega^{-2}} \nabla_a \left(\Omega^{-2} \mathring{K}^c \right)$$

$$= \frac{\Omega^{-2} \mathring{K}^a \nabla_a \left(\Omega^{-2} \mathring{K}^c \right)}{\Omega^{-4}} = 0$$

$$K^a = \Omega^{-2} \mathring{K}^a$$

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Summary

	conformal	physical
metric	$\mathring{g}_{ab}$	$\Omega^2 \mathring{g}_{ab}$
affine parameterization	$\mathring{K}^a$	$\Omega^{-2} \mathring{K}^a$
unit 4-velocity	$\mathring{U}^a$	$\Omega^{-1} \mathring{U}^a$

$$w = g_{ab} U^a K^b$$

$$= \Omega^2 \mathring{g}_{ab} \cdot \Omega^{-1} \mathring{U}^a \cdot \Omega^{-2} \mathring{K}^b$$

$$= \underbrace{\Omega^{-1}}_{\Omega = a} \left(\mathring{g}_{ab} \underbrace{\mathring{U}^a}_{\text{Killing}} \underbrace{\mathring{K}^b}_{\text{affinely parameterized.}} \right) \leftarrow \text{constant}$$

$$\Rightarrow \quad a(\tau)\, w(\tau) = \text{constant}$$

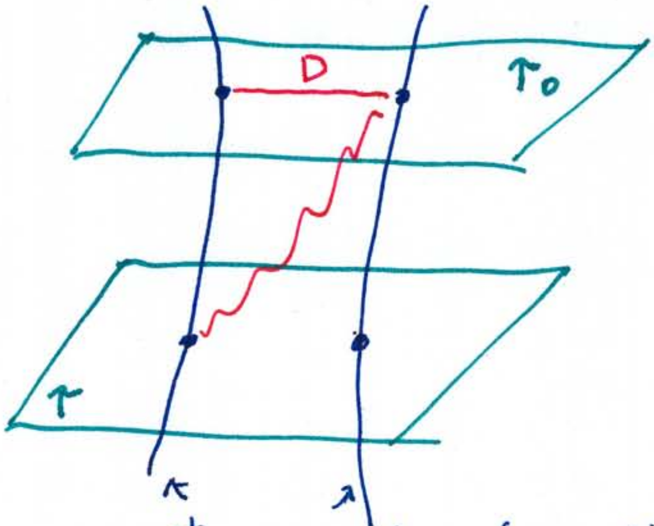
(indep. of τ)

Thus, we find

$$z = \frac{\omega_s}{\omega_r} - 1 = \frac{a_r}{a_s} - 1$$

Distance-Redshift Relation

$$a(\tau) = a_0 + \dot a_0 (\tau - \tau_0) + \frac{1}{2} \ddot a_0 (\tau - \tau_0)^2 + \cdots$$



$\tau < \tau_0$

$\psi = \text{const.}, \ (\theta, \phi)$

$$z(\tau) = \frac{a_0}{a(\tau)} - 1$$

$$= \frac{a_0}{a_0 + \dot a_0 (\tau - \tau_0) + \frac{1}{2} \ddot a_0 (\tau - \tau_0)^2} - 1$$

$$= \frac{1}{1 + \frac{\dot a_0}{a_0}(\tau - \tau_0) + \frac{1}{2}\frac{\ddot a_0}{a_0}(\tau - \tau_0)^2} - 1$$

$$= 1 - \left[\frac{\dot a_0}{a_0}(\tau - \tau_0) + \frac{1}{2}\frac{\ddot a_0}{a_0}(\tau - \tau_0)^2\right]$$

$$+ \left[\frac{\dot a_0}{a_0}(\tau - \tau_0) + \cdots \right]^2$$

$$z(\tau) = \frac{-\dot{a}_0}{a_0}(\tau - \tau_0) + \left[\left(\frac{\dot{a}_0}{a_0}\right)^2 - \frac{\ddot{a}_0}{2a_0}\right](\tau - \tau_0)^2$$

$$+ \cdots$$

H_0, Hubble constant today

$$q_0 := -\frac{a_0 \ddot{a}_0}{\dot{a}_0^2}$$ $\leftarrow$ deceleration parameter

$$z(\tau) = H_0(\tau_0 - \tau) + H_0^2\left[1 + \frac{q_0}{2}\right](\tau_0 - \tau)^2$$

$$+ \cdots$$

But, we want z as a function of <u>distance</u>, not of time, to the emitting galaxy.

Choose the center of coordinates at the reception event.

The geodesic is <u>radial</u>.

$$ds^2 = -d\tau^2 + a^2(\tau)\left[d\chi^2 + \begin{matrix}\sin^2\chi \\ \chi^2 \\ \sinh^2\chi\end{matrix}\, d\Omega^2\right]$$

radial ~~motions~~ null geodesics

$$\boxed{d\tau = a(\tau)\, d\chi}$$

$$\chi = \int_{\tau}^{\tau_0} \frac{d\tau'}{a(\tau')}$$

$$= a_0^{-1} \int_{\tau}^{\tau_0} \frac{a_0}{a(\tau')}\, d\tau'$$

$$= a_0^{-1} \int_{\tau}^{\tau_0} \left[1 + z(\tau')\right] d\tau'$$

$$= a_0^{-1} \int_{\tau}^{\tau_0} \Big[H_0(\tau_0 - \tau') + 1 + H_0^2\left(1 + \frac{q_0}{2}\right)(\tau_0 - \tau')^2\Big]\, d\tau'$$

$$= a_0^{-1} \int_{\tau_0 - \tau}^{0} \left[1 + H_0 x + H_0^2\left(1 + \frac{q_0}{2}\right)x^2\right] \cdot -dx$$

$$= a_0^{-1} \left[(\tau_0 - \tau) + \frac{1}{2} H_0^2 (\tau_0 - \tau)^2\right]$$

$$a_0 r = (\tau_0 - \tau) + \tfrac{1}{2} H_0 (\tau_0 - \tau)^2$$

$$a_0 r = D$$

$$D^2 = (\tau_0 - \tau)^2 + \cdots$$

$$\Rightarrow (\tau_0 - \tau) = D - \tfrac{1}{2} H_0 D^2$$

This means that the redshift of photons from a galaxy <u>currently</u> at distance D is

$$z(D) = H_0 \left[D - \tfrac{1}{2} H_0 D^2 \right] + H_0^2 \left[1 + \frac{q_0}{2} \right] D^2 + \cdots$$

$$= H_0 D + \tfrac{1}{2} H_0^2 (1 + q_0) D^2 + \cdots$$