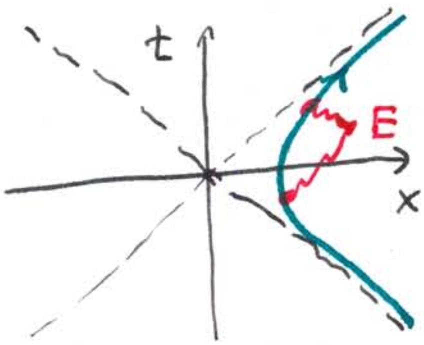
Lecture 5

Gravity and Geometry

Who are the inertial observers?

How do they measure a
<u>non-trivial</u> gravitational field?

Last time: Minkowski spacetime as seen by a uniformly accelerated observer:



$$x(\tau) = \frac{c^2}{g}\cosh\left(\frac{g}{c}\tau\right)$$

$$t(\tau) = \frac{c}{g}\sinh\left(\frac{g}{c}\tau\right)$$

$$\begin{aligned} ds^2 &= -c^2\,dt^2 + dx^2 + dy^2 + dz^2 \\ &= e^{2g\xi/c^2}\left(-c^2\,d\tau^2 + d\xi^2\right) + dy^2 + dz^2 \\ &= -(c^2 + g\chi)\,d\tau^2 + d\chi^2 + dy^2 + dz^2 \end{aligned}$$

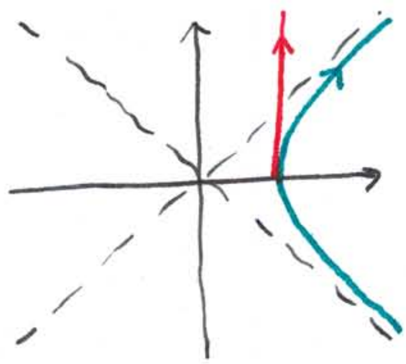
ξ = radio-distance to E (measured with clocks)

χ = metric distance to E (measured with rulers)

Equivalence Principle

A uniformly accelerated observer experiences many of the effects we associate with gravity:

- Normal force
- Dropped objects fall



$$x(t) = \frac{c^2}{g}\sqrt{1+\left(\frac{g}{c}t\right)^2}$$

$$X(t) = \frac{c^2}{g}$$

What does the free-fall curve look like to the accelerated observer?

$$X = \frac{c^2}{g} e^{g\xi/c^2} \cosh\frac{g\tau}{c} = \left(\frac{c^2}{g} + x\right)\cosh\frac{g\tau}{c}$$

$$\| \quad$$

$$\frac{c^2}{g} \rightsquigarrow x(\tau) = \frac{c^2}{g}\left(\operatorname{sech}\frac{g\tau}{c} - 1\right)$$

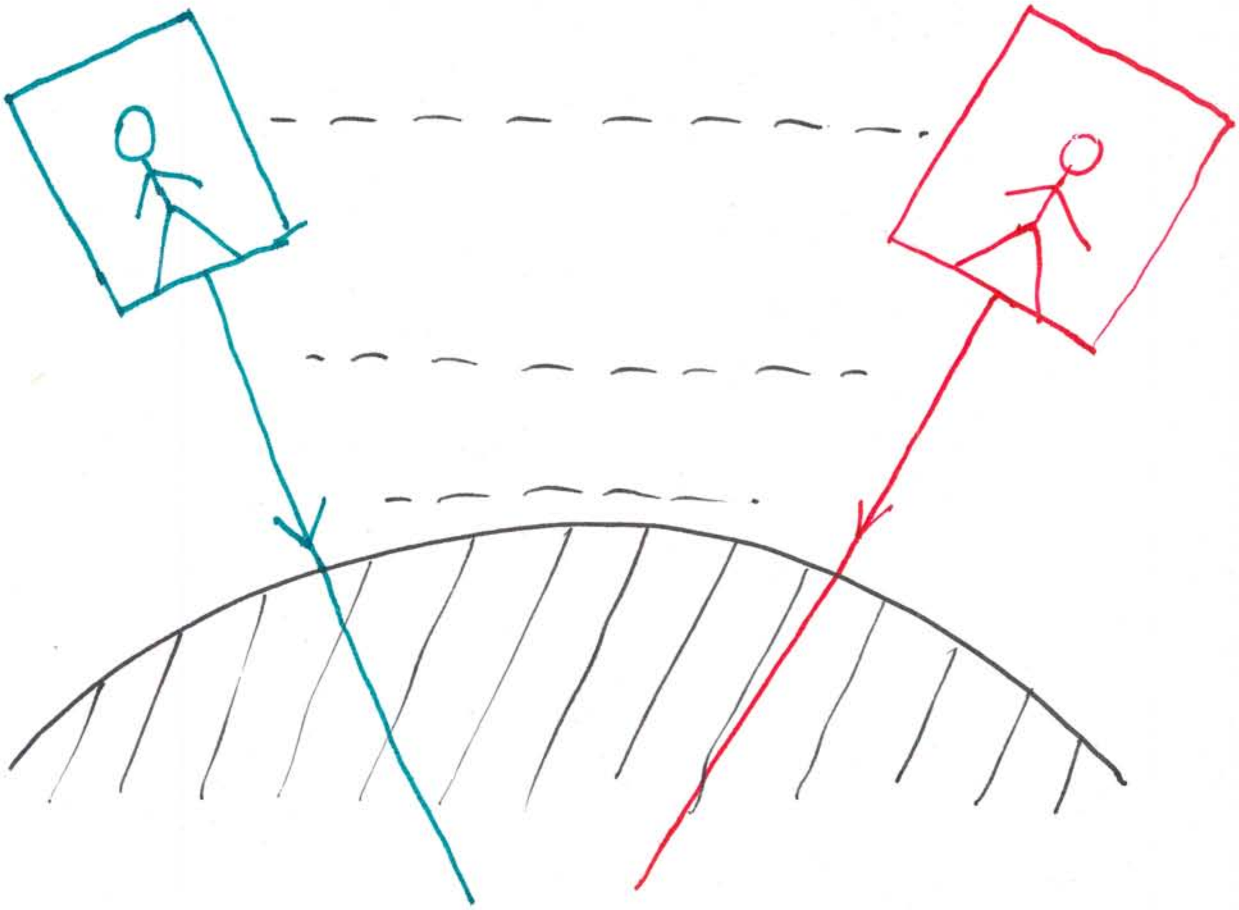
$$= -\frac{1}{2}g\tau^2 + \frac{5}{24}\frac{g^3}{c^2}\tau^4 + \dots$$

On sufficiently short length- and time-scales, uniformly accelerating observers in Minkowski spacetime are like observers in a uniform Newtonian gravitational field.

$$\Delta x \ll \frac{c^2}{g} \qquad \Delta \tau \ll \frac{c}{g}$$

Conversely, on sufficiently short length- and time-scales, a <u>non-accelerating</u> (no normal force) observer in a <u>real</u> gravitational field may believe herself to be in Minkowski spacetime.

> Inertial observers are those in free fall!
>
> (Equivalence Principle)



Gravity <u>focuses</u> free-fall observers.

Inertial observers are <u>local</u>: they must measure things on scales shorter than the focusing effect.

So, on <u>short</u> scales,

- inertial (free-fall) observers move rectilinearly through Euclidean space at uniform relative speeds

⇒ Minkowski metric

⇒ Vector-space structure

On <u>long</u> scales,

- inertial observers are focused and deflected by <u>non-uniform</u> gravitational fields.

⇒ <u>non</u>-Minkowski metric

⇒ <u>no</u> Vector-space structure

General relativity is a theory of a <u>locally</u> Minkowski metric

$$ds^2 = \sum_{\alpha\beta} g_{\alpha\beta}(x)\, dx^\alpha dx^\beta$$

on a spacetime that is <u>locally</u> like $\mathbb{R}^4$. (a <u>manifold</u>)

Gravity and Geometry

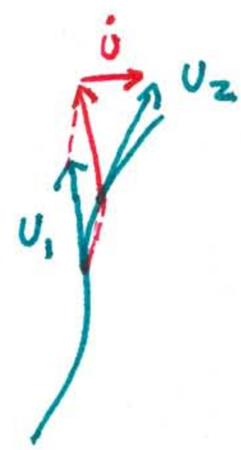
Inertial observers move through locally Minkowski spacetime

$$\frac{d}{d\tau} U^a = 0$$

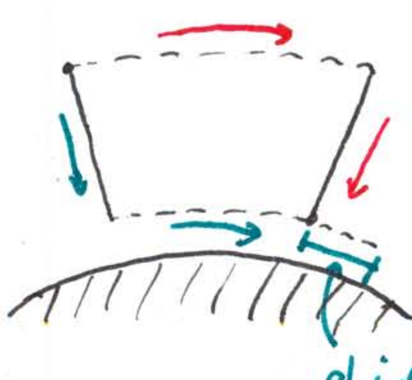
too restrictive; describes parameterized curve:

$$U^a \nabla_a U^b \propto U^b$$

"change of U^b as one moves along U^a"

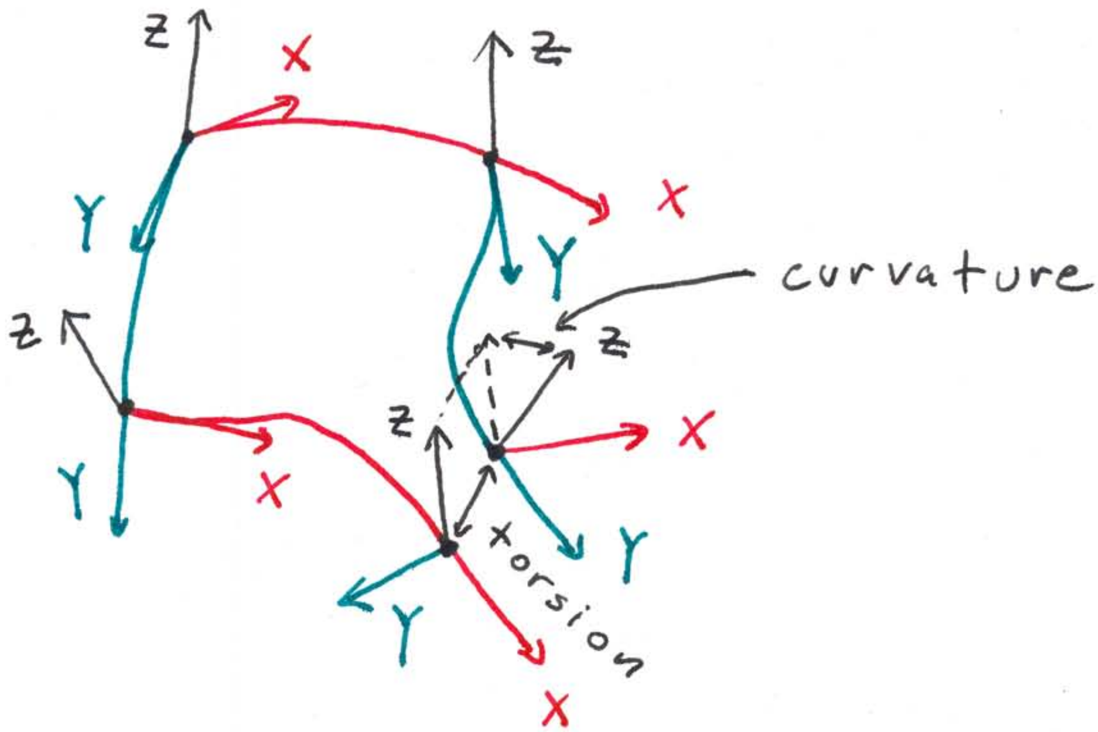


How to measure focusing:



difference!

Consider two observers, each equipped with three gyroscopes:



Observer one follows X then Y.

Observer two follows Y then X.

The gyroscopes keep track of the directions as they move, but respond to gravitational torques: $X^a \nabla_a Y^b = 0$

Knows about gravity

Facts about ∇_a (transport)

gravity

inertial observers

spacetime metric

gravitational torques

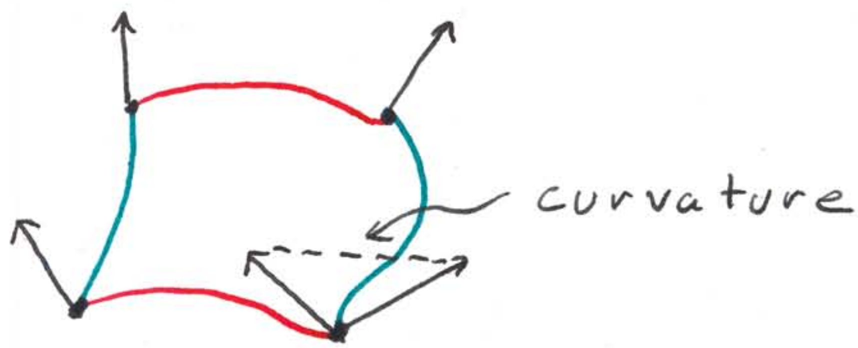
spacetime transport

apparently compatible

(1) When moving vectors from one place to another in spacetime, their lengths and the angles between them (both measured with the local metric) appear to be preserved.

(2) Torsion appears to be a second-order effect.

(Loops close, but transport of vectors around them may be non-trivial.)

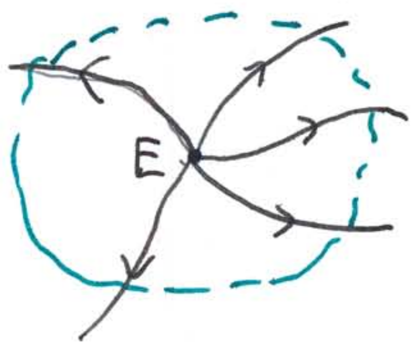


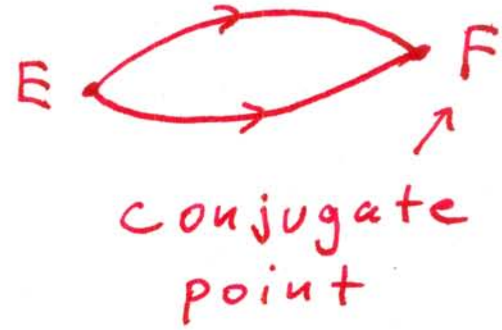
Riemannian Geometry

- metric tensor g_{ab} measures infinitesimal lengths and angles.
- there is a unique transport operator ∇_a that is
  - (a) metric-compatible: $\nabla_a g_{bc} = 0$
  - (b) torsion-free.
- this ∇_a may have curvature

Riemann Normal Coordinates

Draw all <u>non-accelerating</u> curves from a given spacetime event.





Within a sufficiently small neighborhood, there are no conjugate points.

$\Rightarrow$ there is a <u>one-to-one</u> map from initial velocity <u>vectors</u> to surrounding points of spacetime

$$g_{ab}(x) = g_{ab}(\mathring{x}) + R_{acbd}(\mathring{x}) \cdot (x-\mathring{x})^c \cdot (x-\mathring{x})^d + \mathcal{O}\left((x-\mathring{x})^3\right).$$

So, what do we need?

- <u>local</u> vector space structure

  (differentiable manifolds)

- <u>local</u> Minkowski geometry

  (tensor fields)

- transport operation ∇_a

  (derivative operator / affine connection)

- characterize "focusing"

  (Riemann curvature)

- physics.

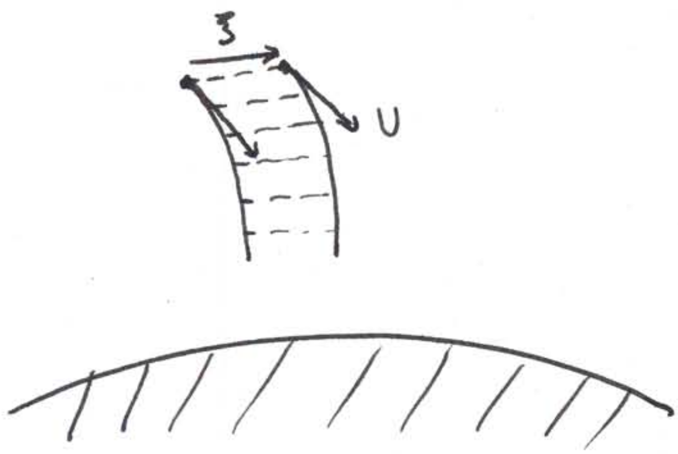
  How do we <u>compute</u> a gravitational field from its source?

  (Einstein field equation)

Geodesic Deviation

non-accelerating curves

focusing



$$\ddot{\xi}^a = (u^c \nabla_c)(u^b \nabla_b)\xi^a$$

$$= -R_{cbd}{}^a \xi^b u^c u^d$$

Riemann curvature <u>is</u> focusing of inertial observers!