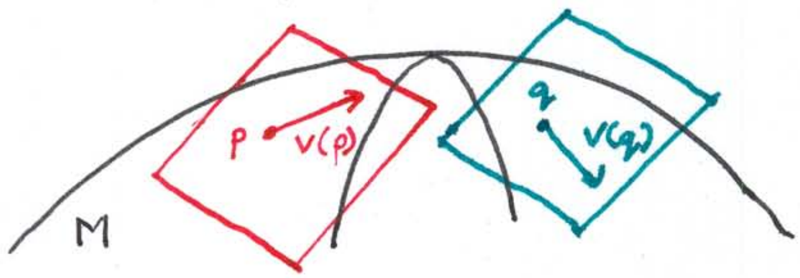
Lecture 8

Tensor Analysis

Vector Fields

A **vector field** assigns to each point $p \in M$ a vector in the tangent space $T_p M$ at that point:



The vector $V(p)$ acts on smooth functions at p to produce a number. Define the function

$$V(f) \rightsquigarrow \underbrace{V(f)(p)}_{\substack{\text{function } V(f) \\ \text{eval. at } p}} := \underbrace{V(p)}_{\substack{\text{vector } V(p) \\ \text{acting on } f}}(f)$$

If the function $V(f)$ is smooth at whenever the function f is, we say that V is **smooth** at p.

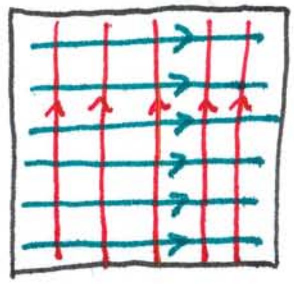
Examples of Vector Fields on $\mathbb{R}^2$

1) $T_x = \frac{\partial}{\partial x}$

2) $T_y = \frac{\partial}{\partial y}$

T_x

T_y



3) $R = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$

$$= R^r \frac{\partial}{\partial r} + R^\theta \frac{\partial}{\partial \theta}$$

$$R^\theta = R(\theta) = x\frac{\partial \theta}{\partial y} - y\frac{\partial \theta}{\partial x}$$

$$\theta = \tan^{-1}\frac{y}{x} \Rightarrow d\theta = \frac{d(y/x)}{1 + (y/x)^2}$$

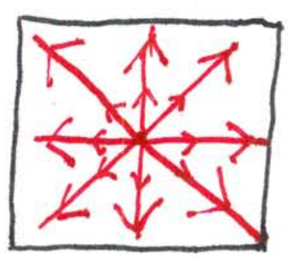
$$= \frac{x\,dy - y\,dx}{x^2 + y^2}$$

$$R^\theta = \frac{(x)^2 + (-y)^2}{x^2+y^2} = 1$$

$$R^r = R(r) = x\frac{\partial r}{\partial y} - y\frac{\partial r}{\partial x} = x\frac{y}{r} - y\frac{x}{r} = 0$$

4) $D = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$

(dilatation)



The Space of Smooth Vector Fields

What structures does the set of all smooth vector fields on a manifold support?

1) Vector Space

$$\underbrace{(\alpha V + \beta W)}_{} (f) := \alpha V(f) + \beta W(f)$$

constants functions

2) Module

$$(f V + g W)(h) := f V(h) + g W(h)$$

functions

(Note: cannot define V/f since $f(p) = 0$ for some $p \in M$.)

3) Lie Bracket (Commutator)

$$[V, W](f) := V(W(f)) - W(V(f))$$

$$V(W(fg)) = V(f W(g) + g W(f))$$

$$= f V(W(g)) + g V(W(f))$$

$$+ V(f) W(g) + V(g) W(f)$$

Lie Brackets of Example Fields

① $[T_x, T_y] = \frac{\partial^2}{\partial x \partial y} - \frac{\partial^2}{\partial y \partial x} = 0$

② $[T_x, R] = \frac{\partial}{\partial x}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) - \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\frac{\partial}{\partial x}$

$$= \frac{\partial}{\partial y} + x\frac{\partial^2}{\partial x \partial y} - y\frac{\partial^2}{\partial x^2}$$

$$- x\frac{\partial^2}{\partial y \partial x} + y\frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial y} = T_y$$

③ $[T_x, D] = \frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) - \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\frac{\partial}{\partial x}$

$$= \frac{\partial}{\partial x} = T_x$$

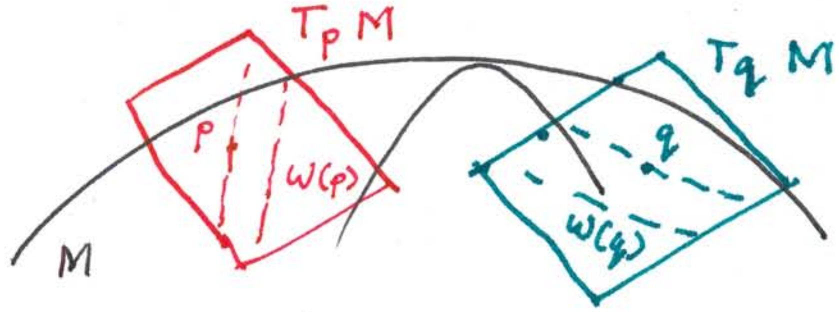
④ $[R, D] = \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)$

$$- \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$= x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x} = 0$$

Smooth Co-Vector Fields

Each tangent space $T_p M$ has a dual space $T_p^* M$. A <u>covector field</u> assigns to each $p \in M$ a dual vector in this space.



The co-vector $\omega(p)$ acts on vectors $V \in T_p M$ to produce a number $\omega(p)(V)$. Given a vector <u>field</u> V, define the <u>function</u>

$$\underbrace{\omega(V)}_{\text{scalar function}}(p) := \omega(p)(V(p))$$

scalar function evaluated at p.

action of $\omega(p)$ on $V(p)$

If $\omega(V)$ is smooth for all smooth vector fields V, we call ω <u>smooth</u>.

Example: <u>Gradient</u>

Let f be a smooth function on M, and define

$$df(V) := V(f)$$ ← smooth function, V acts on f

↑ vector field

↑ gradient of f, co-vector

df is a smooth co-vector field.

<u>Dual to a Coordinate Basis</u>

Last time: Coordinate basis

$$\partial_\alpha(f) := \frac{\partial f_x}{\partial x^\alpha}$$

in each T_pM with $p \in O$.

These are <u>local</u> <u>smooth</u> vector fields ($\partial f_x / \partial x^\alpha$ is smooth.)

$$V = V(x^\alpha)\,\partial_\alpha$$ ← basis expansion

↑ component $V^\alpha = V(x^\alpha) = dx^\alpha(V)$

$\Rightarrow$ dx^α are <u>local</u> smooth dual basis fields on O.

Want to show:

1) dx^α basis for $T_p^* M$

2) dx^α are dual to ∂_α

basis $\rightsquigarrow$ $\omega = \omega_\alpha \, dx^\alpha$

$$\begin{aligned}
\omega(V) &= \omega\left(V^\alpha \partial_\alpha\right) \\
&= V^\alpha \, \omega(\partial_\alpha) \leftarrow := \omega_\alpha \\
&= V(x^\alpha)\, \omega(\partial_\alpha) \\
&= dx^\alpha(V)\, \omega(\partial_\alpha) \\
&= \big[\underbrace{\omega(\partial_\alpha)}_{\omega_\alpha} dx^\alpha\big](V)
\end{aligned}$$

$$dx^\alpha(\partial_\beta) = \partial_\beta(x^\alpha) = \frac{\partial x^\alpha}{\partial x^\beta} = \delta^\alpha_\beta$$

Example:

Expand the "scaled gradient"
$\omega = f\,dg$ (f, g smooth functions)
in the dual basis dx^α:

$$\omega(V) = f\,dg(V)$$
$$:= f\,V(g)$$
$$= f\,V^\alpha \partial_\alpha(g)$$
$$:= f\,\partial_\alpha(g)\,V(x^\alpha)$$
$$= f\,\partial_\alpha(g)\,dx^\alpha(V)$$

$\Rightarrow$ ω and $\underbrace{f\,\partial_\alpha(g)}_{\text{functions}}\,dx^\alpha$ have the same action on <u>any</u> vector field V

dual basis co-vectors

$$\Rightarrow \quad \omega = f\,\partial_\alpha(g)\,dx^\alpha$$
$$= f\,\frac{\partial g}{\partial x^\alpha}\,dx^\alpha$$

Tensor Fields

A *tensor* is a multi-linear map

$$T(V_1, \dots, V_m, \omega^1, \dots, \omega^n) = \#$$

tensor of *type* $\binom{m}{n}$.

The *components* of a tensor:

$$T(\partial_{\alpha_1}, \dots, \partial_{\alpha_m}, dx^{\beta_1}, \dots, dx^{\beta_n})$$

$$=: T_{\alpha_1 \cdots \alpha_m}{}^{\beta_1 \cdots \beta_n}$$

$$\rightsquigarrow T = T_{\alpha_1 \cdots \alpha_m}{}^{\beta_1 \cdots \beta_n}$$

$$dx^{\alpha_1} \otimes \cdots \otimes dx^{\alpha_m} \otimes \partial_{\beta_1} \otimes \cdots \otimes \partial_{\beta_n}$$

Contraction Tensor

$$c(V, \omega) := \omega(V) \leftarrow \delta^a_b$$

components:

$$c^\alpha{}_\beta = c(\partial_\beta, dx^\alpha)$$

$$= dx^\alpha(\partial_\beta)$$

$$= \partial_\beta(x^\alpha)$$

$$= \frac{\partial x^\alpha}{\partial x^\beta} = \delta^\alpha_\beta$$

Metric Tensor

$$g(V, W) = \# = V \cdot W$$

$\rightsquigarrow$ components

$$g_{\alpha\beta} := g(\partial_\alpha, \partial_\beta) = \partial_\alpha \cdot \partial_\beta$$

Tensor Fields

$$\underbrace{T(v^1, \dots, v^m, w_1, \dots, w_n)}_{\text{smooth vectors and co-vectors}} = \underbrace{\text{fcn.}}_{\text{smooth function}}$$

Tensor Algebra

1) addition, scalar multiplication

2) contraction

$$\sum_\alpha T(\partial_\alpha, \underbrace{v^2, \dots v^m}, dx^\alpha, \underbrace{w_2, \dots, w_n})$$

$$T(\tilde{\partial}_\alpha, d\tilde{x}^\alpha) = T(\partial_\gamma, dx^\gamma)$$

$$= T\left(\frac{\partial x^\gamma}{\partial \tilde{x}^\alpha} \partial_\gamma, \frac{\partial \tilde{x}^\alpha}{\partial x^\beta} dx^\beta\right)$$

$$= \underbrace{\frac{\partial x^\gamma}{\partial \tilde{x}^\alpha} \frac{\partial \tilde{x}^\alpha}{\partial x^\beta}}_{\frac{\partial x^\gamma}{\partial x^\beta} = \delta^\gamma_\beta} T(\partial_\gamma, dx^\beta)$$