Geometry Exercises IV

1 We convert the vector field T^a into a density of weight +1 by multiplying through by the scalar density $\sqrt{-g}$. Then, Stokes' theorem (7.19) reads

$$\oint_{\partial\Omega} \sqrt{-g}\, T^a \, dS_a = \int_\Omega \partial_a(\sqrt{-g}\, T^a)\, d\Omega$$

Using the Leibniz property of the derivative on the right, we have

$$\begin{aligned}\partial_a(\sqrt{-g}\, T^a) &= \tfrac{1}{2}(-g)^{-1/2} \cdot -\partial_a g \cdot T^a + \sqrt{-g}\,\partial_a T^a \\ &= -\tfrac{1}{2}(-g)^{-1/2} \cdot 2g\, \Gamma^b_{ba} T^a + \sqrt{-g}\,\partial_a T^a \\ &= \sqrt{-g}\left(\partial_a T^a + \Gamma^b_{ba} T^a\right) = \sqrt{-g}\, \nabla_a T^a\end{aligned}$$

Here, we have used (7.9) to calculate $\partial_a g$ and (6.22) to form the covariant derivative. The result follows immediately.

2 The wedge product (4.10) of three 1-forms generalizes immediately to the p-fold wedge product as the alternating sum of tensor products:

$$\tilde{r}^1 \wedge \dots \wedge \tilde{r}^p := \sum_\pi (-1)^{|\pi|}\, \tilde{r}^{\pi(1)} \otimes \dots \otimes \tilde{r}^{\pi(p)}$$

The sum here is over the permutation group on p symbols.

Any p-form can be written as a linear combination of such elementary wedge products using a basis $\tilde{\omega}^\alpha$. The key point is the total anti-symmetry of its components:

$$\tilde{p} = \sum_\alpha p_{\alpha_1 \cdots \alpha_p} \tilde{\omega}^{\alpha_1} \otimes \cdots \otimes \tilde{\omega}^{\alpha_p}$$

$$= \sum_\alpha p_{[\alpha_1 \cdots \alpha_p]} \tilde{\omega}^{\alpha_1} \otimes \cdots \otimes \tilde{\omega}^{\alpha_p}$$

$$:= \sum_\alpha \frac{1}{p!} \sum_\pi (-1)^\pi p_{\alpha_{\pi(1)} \cdots \alpha_{\pi(p)}} \cdot \tilde{\omega}^{\alpha_1} \otimes \cdots \otimes \tilde{\omega}^{\alpha_p}$$

$$= \frac{1}{p!} \sum_\alpha \sum_\pi (-1)^\pi p_{\alpha_1 \cdots \alpha_p} \cdot \tilde{\omega}^{\alpha_{\pi^{-1}(1)}} \otimes \cdots \otimes \tilde{\omega}^{\alpha_{\pi^{-1}(p)}}$$

$$= \frac{1}{p!} \sum_\alpha p_{\alpha_1 \cdots \alpha_p} \tilde{\omega}^{\alpha_1} \wedge \cdots \wedge \tilde{\omega}^{\alpha_p}$$

Here, we have renamed indices in the penultimate step, and noted that the sum over permutations π in the last step is equal to the sum over their inverses π^{-1}. Thus, we can calculate

$$\tilde{p} \wedge \tilde{q} = \frac{1}{p!\,q!} \sum_\alpha p_{\alpha_1 \cdots \alpha_p} q_{\alpha_{p+1} \cdots \alpha_{p+q}} \tilde{\omega}^{\alpha_1} \wedge \cdots \wedge \tilde{\omega}^{\alpha_{p+q}}$$

$$= \frac{1}{p!\,q!} \sum_\alpha p_{[\alpha_1 \cdots \alpha_p} q_{\alpha_{p+1} \cdots \alpha_{p+q}]} \tilde{\omega}^{\alpha_1} \wedge \cdots \wedge \tilde{\omega}^{\alpha_{p+q}}$$

$$= \frac{1}{(p+q)!} \sum_\alpha (p \wedge q)_{\alpha_1 \cdots \alpha_{p+q}} \tilde{\omega}^{\alpha_1} \wedge \cdots \wedge \tilde{\omega}^{\alpha_{p+q}}$$

The first pair of equalities comes from the definition of the $(p+q)$-fold wedge product of 1-forms and the total antisymmetry of that product in the α indices. The last equality is the result above applied to the

$(p+q)$-form $\tilde{p} \wedge \tilde{q}$. The result then follows by taking components on both sides.

3 We have

$$(\tilde{\beta} \wedge \tilde{\alpha})(\bar{\xi}) := \frac{1}{(p+q-1)!} \xi^i (\beta \wedge \alpha)_{ij\dots k} \tilde{\omega}^j \wedge \dots \wedge \tilde{\omega}^k$$

$$= \frac{1}{(p+q-1)!} \xi^i \frac{(p+q)!}{p!q!} \beta_{[ij\dots l} \alpha_{mn\dots k]} \tilde{\omega}^j \wedge \dots \wedge \tilde{\omega}^k$$

$$:= \frac{p+q}{p!q!} \xi^i \Big(\frac{p}{p+q} \beta_{i[j\dots l} \alpha_{mn\dots k]}$$

$$+ (-1)^p \frac{q}{p+q} \beta_{[j\dots lm} \alpha_{|i|n\dots k]} \Big) \tilde{\omega}^j \wedge \dots \wedge \tilde{\omega}^k$$

$$:= \frac{1}{p!q!} \Big(p\, \beta(\xi)_{[j\dots l} \alpha_{mn\dots k]}$$

$$+ (-1)^p q\, \beta_{[j\dots lm} \alpha(\xi)_{n\dots k]} \Big) \tilde{\omega}^j \wedge \dots \wedge \tilde{\omega}^k$$

$$= \frac{1}{p!q!} \Big(p \cdot (p-1)!\, q!\, \tilde{\beta}(\bar{\xi}) \wedge \tilde{\alpha} + (-1)^p q \cdot p!\, (q-1)!\, \tilde{\beta} \wedge \tilde{\alpha}(\bar{\xi}) \Big)$$

$$= \tilde{\beta}(\bar{\xi}) \wedge \tilde{\alpha} + (-1)^p \tilde{\beta} \wedge \tilde{\alpha}(\bar{\xi})$$

In the third equality, we have noted that, in the average over permutations, there are p places where the i index can appear on β and q on α. In each case, we have used antisymmetry to move the i index to the first position of the form. We have used (4.13) in the fourth equality, and the first expression for $\tilde{p} \wedge \tilde{q}$ from the last problem in the fifth.

4 a) Doing a cofactor expansion on the first row of A gives

$$\det A = \sum_{i=1}^{n} A^{1i} \cdot (-1)^{i} \cdot \det A^{\hat{1}\hat{i}},$$

where $A^{\hat{1}\hat{i}}$ is the cofactor matrix with the first row and the i^{th} column deleted. Meanwhile, we have

$$\varepsilon_{ij\ldots k} A^{1i} A^{2j} \cdots A^{nk}$$
$$:= \sum_{i=1}^{n} A^{1i} \cdot (-1)^{i} \varepsilon^{(\hat{\imath})}_{j\ldots k} A^{2j} \cdots A^{nk}$$

Here, $\varepsilon^{(\hat{\imath})}_{j\ldots k}$ denotes the alternating symbol with $n-1$ indices taking all values from 1 to n except i. The $(-1)^{i}$ arises because

$$\varepsilon_{1\ldots i\ldots n} = (-1)^{i} \varepsilon_{i1\ldots\hat{\imath}\ldots n} = \varepsilon^{(\hat{\imath})}_{1\ldots\hat{\imath}\ldots n}$$

Thus, the determinant and the ε-expression have the same recurrence relation, and the result follows by induction.

b) The tensor

$$B^{ab\ldots c} := \varepsilon_{ij\ldots k} A^{ai} A^{bj} \cdots A^{ck}$$

is totally anti-symmetric and has $B^{1\ldots n} = \det A$. When we contract it with $\varepsilon_{ab\ldots c}$, we get a sum of identical terms over the $n!$ permutations of indices.

c) The orthonormal basis co-vectors $\tilde{\omega}^\alpha$ have expansions

$$\tilde{\omega}^\alpha = \omega^\alpha_i \, \tilde{d}x^i$$

in the coordinate basis. Thus,

$$\tilde{\omega} := \tilde{\omega}^1 \wedge \tilde{\omega}^2 \wedge \cdots \wedge \tilde{\omega}^n$$

$$= \omega^1_i \, \omega^2_j \cdots \omega^n_k \, \tilde{d}x^i \wedge \tilde{d}x^j \wedge \cdots \wedge \tilde{d}x^k$$

The wedge product here is totally anti-symmetric in the indices $i \cdots k$, so

$$\tilde{\omega} = \det(\omega^\alpha_i) \, \tilde{d}x^i \wedge \tilde{d}x^j \wedge \cdots \wedge \tilde{d}x^k$$

Meanwhile, we have

$$g_{ab} = \eta_{\alpha\beta} \, \omega^\alpha_a \, \omega^\beta_b \qquad \text{with } \eta_{\alpha\beta} = \pm 1, \text{ diagonal}$$

because the $\tilde{\omega}^\alpha$ are orthonormal. Taking coordinate components and a determinant,

$$\det(g_{ij}) = \det(\omega^\alpha_i \, \eta_{\alpha\beta} \, \omega^\beta_j)$$

$$= \det(\omega^\alpha_i) \det(\eta_{\alpha\beta}) \det(\omega^\beta_j)$$

$$= \pm \det(\omega^\alpha_i)^2$$

Thus, $\det(\omega^\alpha_i) = \sqrt{|\det(g_{ij})|}$, and the result follows.

5 a) Using the axioms on p. 134, we have

$$d(f\,dg) = df \wedge dg + (-1)^0 f\,ddg = df \wedge dg$$

b) It follows that

$$d\alpha = \frac{1}{p!} d(\alpha_{i\ldots j}\, dx^i \wedge \cdots \wedge dx^j)$$

$$= \frac{1}{p!}\big(d\alpha_{i\ldots j} \wedge dx^i \wedge \cdots \wedge dx^j$$

$$+ \alpha_{i\ldots j}\, ddx^i \wedge \cdots \wedge dx^j - \cdots$$

$$\pm \alpha_{i\ldots j}\, dx^i \wedge \cdots \wedge ddx^j\big)$$

$$= \frac{1}{p!} d\alpha_{i\ldots j} \wedge dx^i \wedge \cdots \wedge dx^j$$

$$= \frac{1}{p!} \partial_k \alpha_{i\ldots j}\, dx^k \wedge dx^i \wedge \cdots \wedge dx^j$$

Meamwhile, we also have

$$d\alpha = \frac{1}{(p+1)!} (d\alpha)_{ki\ldots j}\, dx^k \wedge dx^i \wedge \cdots \wedge dx^j$$

The result follows immediately.

6 We have

$$(\vec{\nabla} \cdot \vec{\nabla} \times \vec{a})\,\tilde{\omega} = d*(\vec{\nabla} \times \vec{a}) = d*(*da) = dda = 0$$

$$\vec{\nabla} \times \vec{\nabla} a = *d(\vec{\nabla} a) = *dda = 0$$

7 A curl-free vector field satisfies

$$0 = \vec{\nabla} \times \vec{a} = * da \quad \Rightarrow \quad da = 0$$

$$\Rightarrow \quad a = db \quad \Rightarrow \quad \vec{a} = \vec{\nabla} b$$

A divergence-free vector field satisfies

$$0 = (\vec{\nabla} \cdot \vec{a})\,\tilde{\omega} = d * a$$

$$\Rightarrow \quad * a = db \quad \Rightarrow \quad a = * db \quad \Rightarrow \quad \vec{a} = \vec{\nabla} \times \vec{b}$$

8 a) The divergence is defined by

$$(\mathrm{div}_{\tilde{\omega}}\, \bar{\xi})\, \tilde{\omega} = \tilde{d}(\tilde{\omega}(\bar{\xi}))$$

But we have

$$\tilde{\omega}(\bar{\xi}) = f \cdot (\tilde{d}x^1 \wedge \cdots \wedge \tilde{d}x^n)(\bar{\xi})$$

$$= \sum_{i=1}^{n} f \cdot (-1)^{i-1}\, \tilde{d}x^1 \wedge \cdots \wedge \tilde{d}x^i(\bar{\xi}) \wedge \cdots \wedge \tilde{d}x^n$$

$$= \sum_{i=0}^{n} f \xi^i (-1)^{i-1}\, \tilde{d}x^1 \wedge \cdots \wedge \widehat{\tilde{d}x^i} \wedge \cdots \wedge \tilde{d}x^n$$

$$\Rightarrow d(\tilde{\omega}(\bar{\xi})) = \sum_{i,j} \partial_j (f \xi^i)(-1)^{i-1}\, \tilde{d}x^j \wedge \tilde{d}x^1 \wedge \cdots \wedge \widehat{\tilde{d}x^i} \wedge \cdots \wedge \tilde{d}x^n$$

The notation $\widehat{\tilde{d}x^i}$ here means that $\tilde{d}x^i$ is excluded in the product. Thus, we must have $j = i$ to get a non-zero result, and we must move $\tilde{d}x^j = \tilde{d}x^i$ past $i-1$ 1-forms in the product The result follows.

8 b) Here, we use the determinant formula

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

$$\rightsquigarrow \det g = \begin{vmatrix} 1 & & \\ & r^2 & \\ & & r^2\sin^2\theta \end{vmatrix} = r^4 \sin^2\theta$$

$$\Rightarrow \tilde{\omega} = |g|^{1/2} dr \wedge d\theta \wedge d\phi = r^2 \sin\theta \, dr \wedge d\theta \wedge d\phi$$

and

$$\operatorname{div} \bar{\xi} = \frac{1}{r^2\sin\theta} \frac{\partial}{\partial r}\left(r^2 \sin\theta \, \xi^r\right) + \frac{1}{r^2\sin\theta} \frac{\partial}{\partial \theta}\left(r^2 \sin\theta \, \xi^\theta\right)$$

$$+ \frac{1}{r^2\sin\theta} \frac{\partial}{\partial \phi}\left(r^2 \sin\theta \, \xi^\phi\right)$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \xi^r\right) + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \, \xi^\theta\right) + \frac{\partial}{\partial \phi} \xi^\phi$$

9 a) We have first of all that

$$(* \bar{\xi})_{j \cdots k} := \frac{1}{1!} \xi^i w_{ij\cdots k} = (\tilde{\omega}(\bar{\xi}))_{j\cdots k}$$

We also have $* w = 1$, so

$$\operatorname{div}_{\tilde{\omega}} \bar{\xi} = * (\operatorname{div}_{\tilde{\omega}} \bar{\xi}) = * \tilde{d} (\tilde{\omega}(\bar{\xi})) = * \tilde{d} * \bar{\xi}$$

b) If F is a p-vector, $*F$ is an $(n-p)$-form, $d * F$ is an $(n-p+1)$-form, and $* d * F$ is an $[n-(n-p+1)]$-vector, which is a $(p-1)$-vector.

To calculate its components, we write

$$(*d*F)^{i\cdots j} := \frac{1}{(n-p+1)!}\,\omega^{k\ell\cdots m i\cdots j}\,(d*F)_{k\ell\cdots m}$$

$$:= \frac{1}{(n-p+1)!}\,\omega^{k\ell\cdots m i\cdots j}\cdot(n-p+1)\,\partial_{[k}(*F)_{\ell\cdots m]}$$

$$:= \frac{1}{(n-p)!}\,\omega^{k\ell\cdots m i\cdots j}\,\partial_k\left(\frac{1}{p!}\,\omega_{ab\cdots c\ell\cdots m}F^{ab\cdots c}\right)$$

$$= \frac{1}{(n-p)!\,p!}\,\omega^{k\ell\cdots m i\cdots j}\,\omega_{ab\cdots c\ell\cdots m}\,\partial_k F^{ab\cdots c}$$

Here, we have used the fact that the components of $\omega_{i\cdots j} = \varepsilon_{i\cdots j}$ are constants ± 1 or 0. We now move the $n-p$ indices $\ell\cdots m$ past the p indices $ab\cdots c$ in the second ω, and past the one index k in the first, and then use ε-δ identities

$$\omega^{k\ell\cdots m i\cdots j}\,\omega_{ab\cdots c\ell\cdots m} =$$

$$= (-1)^{n-p}\,\omega^{\ell\cdots m k i\cdots j}\cdot(-1)^{p(n-p)}\,\omega_{\ell\cdots m ab\cdots c}$$

$$= (-1)^{(n-p)(p+1)}\,\delta^{\ell\cdots m k i\cdots j}_{\ell\cdots m ab\cdots c}$$

$$= (-1)^{(n-p)(p-1)}\,(n-p)!\,\delta^{ki\cdots j}_{ab\cdots c}$$

$$= (-1)^{(n-p)(p-1)}\,(n-p)!\,p!\,\delta^{k}_{[a}\delta^{i}_{b}\cdots\delta^{j}_{c]}$$

Thus, we find

$$(*d*F)^{i\cdots j} = (-1)^{(n-p)(p-1)}\,\delta^{k}_{[a}\delta^{i}_{b}\cdots\delta^{j}_{c]}\,\partial_k F^{ab\cdots c}$$

$$= (-1)^{(n-p)(p-1)}\,\partial_k F^{ki\cdots j}$$

c) In this case, we have

$$w_{i\dots j} = f\,\varepsilon_{i\dots j} \quad \text{and} \quad w^{i\dots j} = f^{-1}\varepsilon^{i\dots j}$$

Making the appropriate changes in the calculation above, we have an extra f inside the derivative and an extra f^{-1} outside. Thus, we find

$$(\operatorname{div}_{\tilde{w}} F)^{i\dots j} = f^{-1}\left(f F^{k i\dots j}\right)_{,k}$$