Physics 6938 Florida Atlantic University
General Relativity Fall, 2007

Problem Set III

Due: Tuesday, 30 October 2007

1. If we neglect material stresses, the stress-energy tensor of a slowly-moving matter distribution can be written, correct to first order in velocity, as

$$T_{ab} = 2\,\hat{t}_{(a}\,p_{b)} + (\hat{t}^c\,p_c)\,\hat{t}_a\,\hat{t}_b,$$

where $p_a := -T_{ab}\,\hat{t}^b$ is the mass-energy current density four-vector and $\hat{t}^a$ is the 4-velocity of a "Newtonian" observer. Let h_{ab} denote the solution of the post-Minkowski field equation in the de Donder gauge for this source. Show that

$$A_a := -\tfrac{1}{4}\,h_{ab}\,\hat{t}^b$$

satisfies the Maxwell equations in Lorentz gauge with source $J_a := p_a$. Furthermore, neglecting time-derivatives of the fields, show that

$$h_{ab} = 4\left[2\,\hat{t}_{(a}\,A_{b)} + (\hat{t}^c\,A_c)\,\hat{t}_a\,\hat{t}_b\right]$$

for such a source.

2. Show that the geodesic equation applied to a perturbed metric of the form found in the previous problem yields an acceleration

$$\mathbf{a} = -\mathbf{E} - 4\mathbf{v}\times\mathbf{B},$$

correct to first order in the velocity of a test mass. Here, $\mathbf{E}$ and $\mathbf{B}$ are respectively the electric and magnetic fields on the "Newtonian" slices of spacetime derived from the 4-vector potential A_a using the standard formulae from electromagnetism.

3. A uniform, rigid, thin shell of radius R and mass $M \ll R$ rotates slowly with angular velocity $\boldsymbol{\omega}$. Show that the electric and magnetic fields of the previous problem are

$$\mathbf{E} = 0 \quad \text{and} \quad \mathbf{B} = \frac{2M}{3R}\,\boldsymbol{\omega}$$

within the shell. An observer at rest of the center of this shell parallel propagates a spatial vector s^a with $s^a\,\hat{t}_a = 0$ along her world-line. Show that the inertial components of s^a precess according to

$$\frac{\mathrm{d}\mathbf{s}}{\mathrm{d}t} = \boldsymbol{\Omega}\times\mathbf{s} \quad \text{with} \quad \boldsymbol{\Omega} = 2\mathbf{B} = \frac{4M}{3R}\,\boldsymbol{\omega},$$

relative to transport in exact Minkoski spacetime. This effect roughly demonstrates the dragging of inertial frames by rotating bodies in general relativity.

4. The geometry outside a large ($R \gg M$), static, spherical source may be written

$$\mathrm{d}s^2 = -\big(1 - \tfrac{2M}{r}\big)\,\mathrm{d}t^2 + \big(1 + \tfrac{2M}{r}\big)\big(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\big),$$

correct to first order, where (t, x, y, z) are the inertial coordinates on the Minkowski background of the inertial observer who sees the source at rest.

a. Let k^a be the tangent to a null geodesic in an affine parameterization, and let $t^a := \partial_t^a$. Show that $e := -t^a\, k_a$ is constant along the geodesic.

b. An atom at rest on the surface of the sun emits a photon of frequency ω_e, which is absorbed by an atom at rest far from the sun. Show that the absorbed photon has a frequency ω_r that is red-shifted by the amount

$$z := \frac{\omega_\mathrm{e} - \omega_\mathrm{r}}{\omega_\mathrm{r}} = \frac{M_\odot}{R_\odot} \approx 2 \times 10^{-6}.$$

Hint: What are the four-velocities u_e^a and u_r^a of the sending a receiving atoms in their proper-time parameterizations?

5. Consider linearized gravity over a *curved* background geometry $\mathring{g}_{ab}$. Show that, under a gauge transformation

$$h_{ab} \mapsto \tilde{h}_{ab} := h_{ab} + 2\,\mathring{\nabla}_{(a}\,\phi_{b)} - \mathring{g}_{ab}\,\mathring{\nabla}_c\,\phi^c,$$

the connection perturbation transforms according to

$$\dot{\nabla}_{ab}{}^c \mapsto \dot{\tilde{\nabla}}_{ab}{}^c := \dot{\nabla}_{ab}{}^c + \phi^m\,\mathring{R}_{mab}{}^c - \mathring{\nabla}_a\,\mathring{\nabla}_b\,\phi^c.$$

Hint: Recall the slightly simpler transformation law for the ordinary metric perturbation $\dot{g}_{ab}$, and note that $\dot{\nabla}_{ab}{}^c$ is linear in $\dot{g}_{ab}$. You will need to use a Bianchi identity.

6. Consider, as in the previous problem, the action of a gauge transformation in linearized gravity over a curved background.

a. Using the result of the previous problem, show explicitly that

$$\dot{R}_{abc}{}^d \mapsto \dot{\tilde{R}}_{abc}{}^d = \dot{R}_{abc}{}^d + \mathscr{L}_\phi\,\mathring{R}_{abc}{}^d.$$

b. Show explicitly, as a result of the previous part, that the Ricci and Einstein tensors transform analogously:

$$\dot{R}_{ab} \mapsto \dot{\tilde{R}}_{ab} := \dot{R}_{ab} + \mathscr{L}_\phi\,\mathring{R}_{ab} \qquad \text{and} \qquad \dot{G}_{ab} \mapsto \dot{\tilde{G}}_{ab} := \dot{G}_{ab} + \mathscr{L}_\phi\,\mathring{G}_{ab}.$$

c. Give *general* arguments, based on the *tensorial* character of $G_{ab}(\lambda)$ and the action of a smooth family $\Phi(\lambda)$ of diffeomorphisms on the spacetime geometry at each λ, to support the latter transformation law above.

d. Use the general arguments of the previous part to show that the source tensor $\dot{T}_{ab}$ must transform according to

$$\dot{T}_{ab} \mapsto \dot{\tilde{T}}_{ab} := \dot{T}_{ab} + \mathscr{L}_\phi\,\mathring{T}_{ab}$$

under a gauge transformation. Thus, show that the the first-order field equation is gauge-covariant even on a non-vacuum background.