Physics 6938 | Florida Atlantic University
General Relativity | Fall, 2007

Problem Set IV

Due: Thursday, 8 November 2007

1. A test particle moves on a circular orbit of radius R in Schwarzschild spacetime.

   a. Show that a static observer O_R at radius R will measure the speed of the orbiting particle to be
   $$v_R = \sqrt{\frac{M}{R-2M}}.$$
   What is the speed of a test particle in the innermost stable circular orbit? What is the speed of a test particle in the innermost unstable circular orbit?

   b. Show that a static observer O_∞ at infinity will measure the speed of the orbiting particle to be
   $$v_\infty = \sqrt{\frac{M}{R}}.$$
   Explain why this result differs from that of the previous part.

2. Let Σ be a spherically symmetric Riemannian three-manifold, and suppose that the gradient $\mathrm{d}r$ of the areal radius is non-zero throughout. Show that an *isotropic radial coordinate* ρ can be defined on Σ in which the metric takes the form
$$\mathrm{d}s^2 = H^2(\rho)\left(\mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\theta^2 + \rho^2\sin^2\theta\,\mathrm{d}\phi^2\right).$$
Thus, every spherically-symmetric three-dimensional space is conformally flat. Write down the Schwarzschild metric in isotropic coordinates.

3. The most general static, axi-symmetric metric in three-dimensional spacetime can be written, at least locally, as
$$\mathrm{d}s^2 = -N^2(r)\,\mathrm{d}t^2 + F^{-2}(r)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2.$$
Show that
$$\mathbf{e}^\alpha = \begin{pmatrix} e^t \\ e^r \\ e^\theta \end{pmatrix} = \begin{pmatrix} N\,\mathrm{d}t \\ F^{-1}\,\mathrm{d}r \\ r\,\mathrm{d}\theta \end{pmatrix}$$
is an orthonormal basis of co-vector fields for this metric, and that the corresponding Cartan matrix takes the form
$$\boldsymbol{\omega}_\alpha{}^\beta = \begin{pmatrix} 0 & -FN'\,\mathrm{d}t & 0 \\ -FN'\,\mathrm{d}t & 0 & -F\,\mathrm{d}\theta \\ 0 & F\,\mathrm{d}\theta & 0 \end{pmatrix}.$$
The order of rows and columns here is the same as that used in the matrix expression of the basis above.

4. Use the result of the previous exercise to show that the Riemann tensor for a three-dimensional, static, axi-symmetric spacetime may be written as

$$\boldsymbol{R}_\alpha{}^\beta = \begin{pmatrix} 0 & \left(FN'\right)' \mathrm{d}t \wedge \mathrm{d}r & F^2 N' \,\mathrm{d}t \wedge \mathrm{d}\theta \\ \left(FN'\right)' \mathrm{d}t \wedge \mathrm{d}r & 0 & -F' \,\mathrm{d}r \wedge \mathrm{d}\theta \\ F^2 N' \,\mathrm{d}t \wedge \mathrm{d}\theta & F' \,\mathrm{d}r \wedge \mathrm{d}\theta & 0 \end{pmatrix},$$

where the concrete indices refer to the orthonormal basis given above. Derive from this the expressions

$$R_{ab} = \frac{F}{Nr}\left[\left(rFN'\right)' e_a^t\, e_b^t - \left(\left(rFN'\right)' + NF' - FN'\right) e_a^r\, e_b^r - \left(FN\right)' e_a^\theta\, e_b^\theta\right]$$

and

$$G_{ab} = \frac{F}{Nr}\left[-NF'\, e_a^t\, e_b^t + FN'\, e_a^r\, e_b^r + r\left(FN'\right)' e_a^\theta\, e_b^\theta\right]$$

for the Ricci and Einstein tensors.

5. The vacuum Einstein equations *with a negative cosmological constant* $\Lambda = -1/\ell^2$ are

$$G_{ab} + \Lambda\, g_{ab} = 0.$$

Show that a static, axi-symmetric vacuum solution of these equations in three-dimensional spacetime has the form

$$\mathrm{d}s^2 = -\left(\frac{r^2}{\ell^2} - M\right)\mathrm{d}t^2 + \left(\frac{r^2}{\ell^2} - M\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2,$$

where M is an arbitrary constant of integration. This is called the (non-rotating) *Bañados–Teitelboim–Zanelli*, or just the *BTZ, metric*.

6. In the non-rotating BTZ metric, show that the scalar curvature invariant $I := R_{abcd}\, R^{abcd}$ takes the constant value $12/\ell^4$.

   a. For negative $M = -\mu^2$, show that the non-rotating BTZ metric has only a time-like conical singularity at the origin, which disappears if and only of $M = -1$. (This critical value of M defines the three-dimensional *anti-de Sitter solution*.)

   b. For positive $M = \mu^2$, show that the non-rotating BTZ metric has only a space-like conical singularity at the origin, which is hidden behind a horizon at $r = \mu\ell$.

   Hint: In part (a), let $\rho(r)$ be the proper length of a radial geodesic from the origin to a point on the circle with circumference $2\pi r$. Show that the circumference $C(\rho)$ of that circle satisfies $C(\rho) = 2\pi\mu\ell \sinh\rho/\ell$.