Problem set IV

1 a) Here, we recall that the Minkowski product of two (normalized) four-velocities is minus the γ-factor for the relative (spatial) speed:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \gamma \\ \gamma \vec{v} \end{pmatrix} = -\gamma$$

In this case, the static observer's 4-velocity is

$$\hat{v}_s(R) = \left(1 - \frac{2M}{R}\right)^{-1/2} \partial_t$$

The orbiting particles 4-velocity is

$$\hat{v}_c(R) = \dot{t}\,\partial_t + \dot{\phi}\,\partial_\phi$$

$$= \left(1 - \frac{2M}{R}\right)^{-1} E\,\partial_t + R^{-2} L\,\partial_\phi$$

Thus, we have

$$-\gamma = -(1-v^2)^{-1/2} = -\left(1-\frac{2M}{R}\right)\cdot\left(1-\frac{2M}{R}\right)^{-1/2}\cdot\left(1-\frac{2M}{R}\right)^{-1} E$$

$$\Rightarrow 1 - v^2 = \left(1 - \frac{2M}{R}\right) E^{-2}$$

But for a circular orbit, we have

$$\frac{1}{2} E^2 = \frac{1}{2}\left(1 - \frac{2M}{R}\right)\left(\frac{L^2}{R^2} + 1\right)$$

$$\Rightarrow 1 - v^2 = \left(\frac{L^2}{R^2} + 1\right)^{-1}$$

Finally, the equilibrium condition give

$$MR^2 - L^2 R + 3ML^2 = 0$$

$$\Rightarrow \quad L^2 = \frac{MR^2}{R-3M}$$

$$\Rightarrow \quad 1 - v^2 = \left(\frac{M}{R-3M} + 1\right)^{-1} = \frac{R-3M}{R-2M} = 1 - \frac{M}{R-2M}$$

The result follows. When $R \to 3M$, the orbital speed approaches unity, the speed of light.

b) Here, we write

$$v_\infty = R\frac{d\phi}{dt} = R\frac{\dot{\phi}}{\dot{t}} = \frac{L}{R}\left(1 - \frac{2M}{R}\right)E^{-1}$$

$$\Rightarrow \quad v_\infty^2 = \left(1 - \frac{2M}{R}\right)^2 \frac{L^2}{R^2} E^{-2} = \left(1 - \frac{2M}{R}\right)\frac{L^2}{R^2}\left(\frac{L^2}{R^2} + 1\right)^{-1}$$

$$= \left(1 - \frac{2M}{R}\right)\left(1 + \frac{R^2}{L^2}\right)^{-1} = \left(1 - \frac{2M}{R}\right)\left(1 + \frac{R-3M}{M}\right)^{-1}$$

$$= \left(1 - \frac{2M}{R}\right)\frac{M}{R-2M} = \frac{M}{R}$$

The difference results from the time dilation by gravitational redshift.

2 We proved in class that this metric can be written in terms of the areal radius as

$$ds^2 = F^{-2}(r)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$= H^2(\rho)\,d\rho^2 + H^2(\rho)\,\rho^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Comparing coefficients, we clearly must have

$$r = H(\rho)\,\rho \qquad \text{and} \qquad F^{-1}(r)\,dr = H(\rho)\,d\rho$$

$$\Rightarrow \frac{dr}{r F(r)} = \frac{d\rho}{\rho} \quad \Rightarrow \quad \ln \rho = \int^r \frac{dr'}{r' F(r')}$$

$$\Rightarrow \rho(r) = e^{\int^r \frac{dr'}{r' F(r')}}$$

$$\Rightarrow H(\rho(r)) = \frac{r}{\rho} = r\, e^{-\int^r \frac{dr'}{r' F(r')}}$$

Note that these indefinite integrals give ρ and H up to complementary scalings, which moreover leave ds^2 invariant.

For Schwarzschild, we must calculate

$$\int^r \frac{dr'}{r' F(r')} = \int^r \frac{dr'}{r' \sqrt{1 - 2M/r'}} = \int^r \frac{dr'}{\sqrt{r'^2 - 2Mr'}}$$

$$= \int^r \frac{dr'}{\sqrt{(r'-M)^2 - M^2}} = \ln\left(r - M + \sqrt{(r-M)^2 - M^2}\right) + c$$

$$\Rightarrow \rho = C\left[r - M + \sqrt{(r-M)^2 - M^2}\right]$$

$$\Rightarrow \left[C^{-1}\rho - (r-M)\right]^2 = (r-M)^2 - M^2$$

$$\Rightarrow C^{-2}\rho^2 + M^2 = 2C^{-1}\rho(r-M)$$

$$\Rightarrow \frac{(C^{-1}\rho + M)^2}{2C^{-1}\rho} = r$$

$$\Rightarrow H(\rho) = \frac{r}{\rho} = \frac{(C^{-1}\rho + M)^2}{2C^{-1}\rho^2} = \frac{1}{2C}\left(1 + \frac{CM}{\rho}\right)^2$$

Demanding $\rho \sim r$ at infinity sets $C = \frac{1}{2}$, reproducing (14.58) from the book.

3 First, we calculate

$$\eta_{\alpha\beta}\, de^{\beta} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} N' dr \wedge dt \\ 0 \\ dr \wedge d\theta \end{pmatrix} = \begin{pmatrix} N' dt \wedge dr \\ 0 \\ dr \wedge d\theta \end{pmatrix}$$

Setting this equal to $-\omega_{\alpha\beta} \wedge e^{\beta}$ lets us read off several terms in the Cartan matrix because its diagonal entries must vanish by anti-symmetry:

$$-\omega_{\alpha\beta} = \begin{pmatrix} 0 & FN' dt & 0 \cdot dt \\ 0 \cdot dr & 0 & 0 \cdot dr \\ 0 \cdot d\theta & -F d\theta & 0 \end{pmatrix}$$

We have suppressed all undetermined terms. Now imposing anti-symmetry gives

$$-\omega_{\alpha\beta} = \begin{pmatrix} 0 & & \\ -FN' dt + 0 \cdot dr & 0 & \\ 0 \cdot dt + 0 \cdot d\theta & -F d\theta + 0 \cdot dr & 0 \end{pmatrix}$$

There are remarkably few undetermined terms, and a direct calculation shows they vanish:

$$\omega_{\alpha}{}^{\beta} = \omega_{\alpha\gamma} \cdot \eta^{\gamma\beta} = \begin{pmatrix} 0 & -FN' dt & 0 \\ FN' dt & 0 & -F d\theta \\ 0 & F d\theta & 0 \end{pmatrix} \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -FN' dt & 0 \\ -FN' dt & 0 & -F d\theta \\ 0 & F d\theta & 0 \end{pmatrix}$$

4 The two terms in the Riemann tensor are

$$d\omega_\alpha{}^\beta = \begin{pmatrix} 0 & -(FN')'dr\wedge dt & 0 \\ -(FN')'dr\wedge dt & 0 & -F'dr\wedge d\theta \\ 0 & F'dr\wedge d\theta & 0 \end{pmatrix}$$

$$(\omega\wedge\omega)_\alpha{}^\beta = \begin{pmatrix} 0 & 0 & F^2N'dt\wedge d\theta \\ 0 & 0 & 0 \\ -F^2N'd\theta\wedge dt & 0 & 0 \end{pmatrix}$$

$$\Rightarrow R_\alpha{}^\beta = \begin{pmatrix} 0 & (FN')'dt\wedge dr & F^2N'dt\wedge d\theta \\ (FN')'dt\wedge dr & 0 & -F'dr\wedge d\theta \\ F^2N'dt\wedge d\theta & F'dr\wedge d\theta & 0 \end{pmatrix}$$

The Ricci tensor is given by

$$R_\alpha = R_\alpha{}^\beta \cdot e_\beta = R_\alpha{}^\beta \begin{pmatrix} N^{-1}\partial_t \\ F\,\partial_r \\ r^{-1}\partial_\theta \end{pmatrix}$$

$$= \begin{pmatrix} (FN')'F\,dt + F^2N'r^{-1}dt \\ -(FN')'N^{-1}dr - F'r^{-1}dr \\ -F^2N'N^{-1}d\theta - F'F\,d\theta \end{pmatrix}$$

$$= \begin{pmatrix} FN^{-1}r^{-1}\left[(FN')'r + FN'\right]e^t \\ -FN^{-1}r^{-1}\left[(FN')'r + NF'\right]e^r \\ -FN^{-1}r^{-1}\left[FN' + NF'\right]e^\theta \end{pmatrix}$$

The expression for Ricci follows directly. To get Eintein, we must calculate

$$R = \frac{F}{Nr}\left[-(rFN')' - \left((rFN')' + NF' - FN'\right) - (FN)'\right]$$

This simplifies to

$$R = -2\,\frac{F}{Nr}\left[(rFN')' + NF'\right]$$

$$\Rightarrow G_{ab} = R_{ab} - \tfrac{1}{2} R\, g_{ab}$$

$$= R_{ab} + \frac{F}{Nr}\left[(rFN')' + NF'\right]\left[-e^t_a e^t_b + e^r_a e^r_b + e^\theta_a e^\theta_b\right]$$

$$= \frac{F}{Nr}\left[-NF' e^t_a e^t_b + FN' e^r_a e^r_b + \left((rFN')' - FN'\right) e^\theta_a e^\theta_b\right]$$

The result follows.

5 The field equation is

$$G_{ab} + \Lambda g_{ab} = G_{ab} - \frac{1}{\ell^2}\left[-e^t_a e^t_b + e^r_a e^r_b + e^\theta_a e^\theta_b\right]$$

$$= -\frac{1}{r}\left(FF' - \frac{r}{\ell^2}\right) e^t_a e^t_b + \frac{1}{r}\left(F^2 \frac{N'}{N} - \frac{r}{\ell^2}\right) e^r_a e^r_b$$

$$+ \left(\frac{F}{N}(FN')' - \frac{1}{\ell^2}\right) e^\theta_a e^\theta_b = 0$$

The tt-component gives

$$FF' = \frac{r}{\ell^2} \quad \Rightarrow \quad \tfrac{1}{2} F^2 = \tfrac{1}{2}\frac{r^2}{\ell^2} - \tfrac{1}{2} M$$

for a suitable constant M. The rr-component then gives

$$\frac{N'}{N} = F^{-2}\frac{r}{\ell^2} = \frac{F'}{F} \quad \Rightarrow \quad N' = CF$$

for a suitable constant C, which may be absorbed into the time coordinate t, so $C = 1$.

We must check the remaining $\theta\theta$ equation:

$$\frac{F}{N}(FN')' - \frac{1}{\ell^2} = \left(\tfrac{1}{2}F^2\right)'' - \frac{1}{\ell^2} = \frac{1}{2}\left(\frac{r^2}{\ell^2} - M\right)'' - \frac{1}{\ell^2} = 0$$

Thus, the BTZ metric solves all equations.

6 We have, given $F^2 = N^2 = \frac{r^2}{\ell^2} - M$

$$R_\alpha{}^\beta = \begin{pmatrix} 0 & (\tfrac{1}{2}F^2)''\, dt \wedge dr & F(\tfrac{1}{2}F^2)'\, dt \wedge d\theta \\ (\tfrac{1}{2}F^2)''\, dt \wedge dr & 0 & -F^{-1}(\tfrac{1}{2}F^2)'\, dt \wedge d\theta \\ F(\tfrac{1}{2}F^2)'\, dt \wedge d\theta & F^{-1}(\tfrac{1}{2}F^2)'\, dr \wedge d\theta & 0 \end{pmatrix}$$

$$= \frac{1}{\ell^2}\begin{pmatrix} 0 & dt \wedge dr & rF\, dt \wedge d\theta \\ dt \wedge dr & 0 & -F^{-1} r\, dr \wedge d\theta \\ rF\, dt \wedge d\theta & F^{-1} r\, dr \wedge d\theta & 0 \end{pmatrix}$$

$$= \frac{1}{\ell^2}\begin{pmatrix} 0 & e^t \wedge e^r & e^t \wedge e^\theta \\ e^t \wedge e^r & 0 & -e^r \wedge e^\theta \\ e^t \wedge e^\theta & e^r \wedge e^\theta & 0 \end{pmatrix}$$

The curvature invariant is

$$I = -R_{ab\alpha}{}^\beta R^{ab}{}_\beta{}^\alpha = -\mathrm{Tr}(R \cdot R)$$

$$= \frac{-1}{\ell^4} \cdot 2\left([e^t \wedge e^r]^2 + [e^t \wedge e^\theta]^2 - [e^r \wedge e^\theta]^2\right)$$

$$= \frac{-2}{\ell^4}(-2-2-2) = \frac{12}{\ell^4}$$

Thus, there is no curvature singularity anywhere in the BTZ spacetime.

8

a) When $M=-\mu^2$, the metric

$$ds^2 = -\left(\frac{r^2}{\ell^2}+\mu^2\right)dt^2 + \left(\frac{r^2}{\ell^2}+\mu^2\right)^{-1}dr^2 + r^2 d\theta^2$$

is regular down to the origin. We find the conical singularity by measuring the proper radius of the circle with coordinate radius r:

$$s(r) = \int_0^r \left(\frac{r'^2}{\ell^2}+\mu^2\right)^{-1/2} dr' = \int_0^{r/\ell} \frac{\ell\, dx}{\sqrt{x^2+\mu^2}}$$

$$= \ell \sinh^{-1}\frac{x}{\mu}\Big|_0^{r/\ell} = \ell \sinh^{-1}\frac{r}{\mu\ell}$$

$$\Rightarrow C(s) = 2\pi r = 2\pi\mu\ell \sinh\frac{s}{\ell}$$

In the limit $s\to 0$, this becomes $2\pi\mu s$, which shows a conical deficit unless $\mu = 1$ and therefore $M=-1$.

b) When $M=\mu^2$, the Key thing is to show that the metric can be continued through the coordinate singularity at $r=\mu\ell$. We do this using a slightly different approach than that taken in class for the Schwarzschild metric. First, as before, we focus on the metric in the tr-plane and write

$$ds^2 = \left(\frac{r^2}{\ell^2}-\mu^2\right)\left[-dt^2 + \left(\frac{r^2}{\ell^2}-\mu^2\right)^{-2}dr^2\right]$$

$$\rightsquigarrow\ dr_* = \left(\frac{r^2}{\ell^2}-\mu^2\right)^{-1}dr = \frac{\ell^2 dr}{r^2-\mu^2\ell^2}$$

$$= \ell^2\, d\left(\frac{1}{2\mu\ell}\ln\frac{r-\mu\ell}{r+\mu\ell}\right)$$

Thus, for $\mu\ell < r < \infty$, we have

$$-\infty < r_* := \frac{\ell}{2\mu} \ln \frac{r-\mu\ell}{r+\mu\ell} < 0.$$

Next, we define the null coordinate $v := t + r_*$, so that $t = v - r_*$, and find

$$ds^2 = \left(\frac{r^2}{\ell^2} - \mu^2\right)\left[-dt^2 + dr_*^2\right] = \left(\frac{r^2}{\ell^2} - \mu^2\right)\left[-dv^2 + 2dv\,dr_*\right]$$

$$= -\left(\frac{r^2}{\ell^2} - \mu^2\right) dv^2 + 2\,dv\,dr$$

This metric is clearly extensible down to $r = 0$. Moreover, the interior metric is isometric to the interior ($r < \mu\ell$) BTZ metric:

$$ds^2 = \left(\mu^2 - \frac{r^2}{\ell^2}\right) dv \left[dv + 2\left(\mu^2 - \frac{r^2}{\ell^2}\right)^{-1} dr\right]$$

$$\left(\mu^2 - \frac{r^2}{\ell^2}\right)^{-1} dr = \frac{\ell^2\, dr}{\mu^2\ell^2 - r^2} = \ell^2\, d\left(\frac{1}{2\mu\ell} \ln \frac{\mu\ell + r}{\mu\ell - r}\right)$$

$$= -d\left(\frac{\ell}{2\mu} \ln \frac{\mu\ell - r}{\mu\ell + r}\right) =: -dr_*$$

We choose the sign here so that $0 < r < \mu\ell$ implies $0 > r_* > -\infty$, and $r_* = -\infty$ again at the horizon. Then, define the coordinate $t := v - r_*$, so that $v = t + r_*$, and

$$ds^2 = \left(\mu^2 - \frac{r^2}{\ell^2}\right) dv\left[dv - 2dr_*\right] = \left(\mu^2 - \frac{r^2}{\ell^2}\right)\left[dt^2 - dr_*^2\right]$$

$$= \left(\mu^2 - \frac{r^2}{\ell^2}\right) dt^2 - \left(\mu^2 - \frac{r^2}{\ell^2}\right)^{-1} dr^2$$

This, of course, is just the same BTZ metric again, now in the interior.

It is debatable whether the problem with the BTZ metric at $r=0$ should be called a conical singularity. I probably shouldn't have asked this. But the ideas are certainly related. Consider the metric on a surface of constant t near $r=0$:

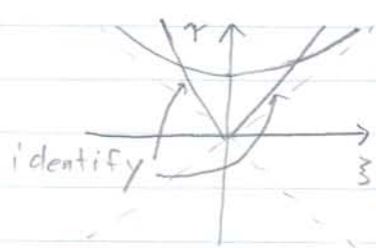
$$ds^2 \stackrel{\sim}{=} -\mu^2 dr^2 + r^2 d\phi^2$$

$$= -[d(\mu r \cosh \tfrac{\phi}{\mu})]^2 + [d(\mu r \sinh \tfrac{\phi}{\mu})]^2$$

Thus, the short-distance geometry near $r=0$ is Minkowski, as we expect. But remember that ϕ is an angular coordinate in reality, so we must identify $\phi=-\pi$ with $\phi=\pi$ (say). If we define

$$\Upsilon := \mu r \cosh \tfrac{\phi}{\mu}$$

$$\xi := \mu r \sinh \tfrac{\phi}{\mu}$$



we can make the map into 2-d Minkowski spacetime explicit. Clearly, the spacetime structure at the origin, the vertex of the identified lines, is atypical. It does not have the standard manifold structure, even when $\mu \to 0$ so that the identification joins the two branches of the light cone. This is what we mean by a conical singularity in a Euclidean manifold, so perhaps the nomenclature isn't too bad. But its subtle, and I still shouldn't have asked. Free points.